AD-778 027

ATMOSPHERIC PRESSURE GAS LASERS

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Prepared for:

Advanced Research Projects Agency

April 1973

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SEMIANNUAL TECHNICAL REPORT ON ATMOSPHERIC PRESSURE GAS LASERS

covering the period
June 1, 1973-November 30, 1973

submitted by

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Sponsored by

Advanced Research Projects Agency
ARPA Order No. 0675, Amend. No. 19

Contract Number: DAHC04-72-C-0044

Program Code No.
62301E

Principal Investigator
Hermann A. Haus
617-253-2585

Contractor:
Massachusetts Institute of Technology
Cambridge, MA 02139

Effective Date of Contract:
June 1, 1972

Short Title of Work:
Atmospheric-Pressure Gas Lasers

Expiration Date:
May 31, 1974

Date of Report:
April, 1973

Amount of Contract:
$80,000

Semiannual Technical Report on

Atmospheric Pressure Gas Lasers

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June 1, 1973-November 30, 1974

SUMMARY

I. Amplification of Two High-Intensity Nanosecond TEA CO_2 Laser Pulses

Yongyut Manichaikul studied the effect of vibrational relaxation upon the replenishment of the upper laser level when two high intensity pulses are passed through a CO_2 TEA amplifying medium. He showed that with increasing pressure in the amplifying medium, the large signal gain of the second pulse increases until it reaches a plateau corresponding to partial recovery of the laser inversion due to redistribution of the vibrational population associated with the asymmetric stretching mode. The experiment indicated that the 001 level of CO_2 was repopulated by the higher 0nm levels at the rate that was directly proportional to the CO_2 partial pressure.

Details of the experiment are described in Appendix I, which is

a reprint of the RLE Quarterly Progress Report No. 111 of October 15, 1973.

II. Mode Locking

C. P. Ausschnitt has begun experimental work aimed at a better understanding of active and passive mode locking of CO_2 lasers. Concurrently, work is in progress to obtain closed-form expressions for mode-locked pulses produced both actively and passively. The literature on the subject is very large; yet in spite of the vast amount of work on the subject, there still is no compact simple theory that would both predict the steady-state mode-locking solutions, and also be amenable to the study of the stability of such steady-state solutions, their transient behavior, their susceptibility to noise, and their stability.

Considerable progress has been made in an effort to accomplish this program. A recent paper by Kurokawa on injection locking provided the basis for an analysis which is both simpler and more flexible than analyses presented heretofore. A paper on the theory of forced mode locking forms Appendix II. While the steady-state solutions worked out in this paper have been obtained previously in the literature, we present here, for the first time, the solution of the forced-mode-locking problem of an inhomogeneously broadened medium and study the transient behavior of the buildup of mode-locked pulses, as well as the stability of the various mode-locked "super

modes", both for the homogeneously and inhomogeneously broadened laser medium. Work is currently in progress that utilizes the same formalism to analyze saturable absorber mode locking.

III. Closed-form Analysis of Electronic Distribution and Pumping Rates

The work carried out on this topic has appeared in the February 1974 issue of the Journal of Applied Physics; pp. 781-791.

APPENDIX I

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Cambridge, Massachusetts 02139

Reprinted from

Quarterly Progress Report No. 111, October 15, 1973

3. P. K. Cheo and R. L. Abrams, "Rotational Relaxation Rate of CO_2 Laser Levels," Appl. Phys. Letters 14, 47-49 (1969).

4. K. F. Herzfeld and T. A. Litovitz, Absorption and Dispersion of Ultrasonic Waves (Academic Press, Inc., New York, 1959).

2. AMPLIFICATION OF TWO HIGH-INTENSITY NANOSECOND TEA CO_2 LASER PULSES (AHN)

National Science Foundation (Grant GK-37979X)

U. S. Army – Research Office – Durham (Contract DAHC04-72-C-0044)

Y. Manichaikul

Experiment

We have previously reported on the generation and amplification of high-intensity nanosecond pulses.[1] Two or three of these pulses were produced. They were from the P(16) transition, 2 ns wide (FWHM), separated by 12 ns. When these pulses were focused into a three-electrode laser amplifier as shown in Fig. VI-12, a peak intensity of 2-3 MW/cm^2 was obtained. A beam splitter was used so that the intensity of the pulses could be monitored. The input and output detectors were as shown in Fig. VI-12. In this experiment the detected input signals were delayed 100 ns by using 60 ft of

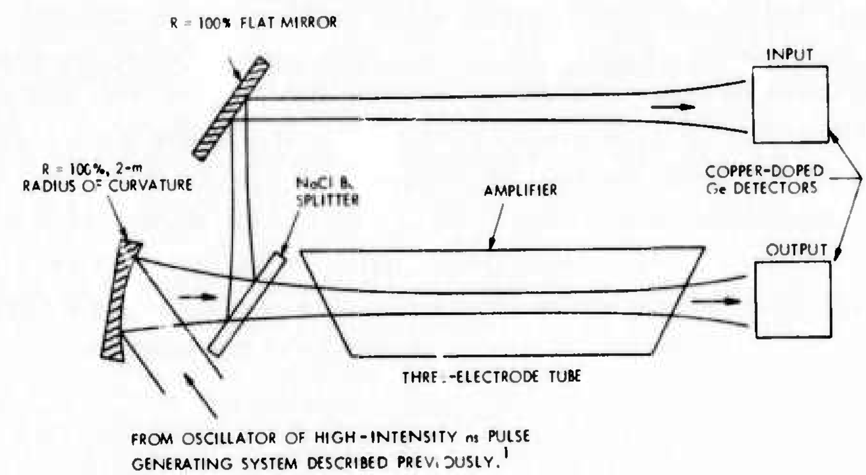

Fig. VI-12. Experimental arrangement for amplification of high-intensity ns pulses. (See Y. Manichaikul.[1])

RG-8 cable. The add mode of a Tektronix oscilloscope was used to display the signals for both input and output pulses on the same screen. The two detectors were calibrated against each other by comparing the oscilloscope picture of the input and output pulses without discharge exciting the three-electrode laser amplifier. Figure VI-13a shows

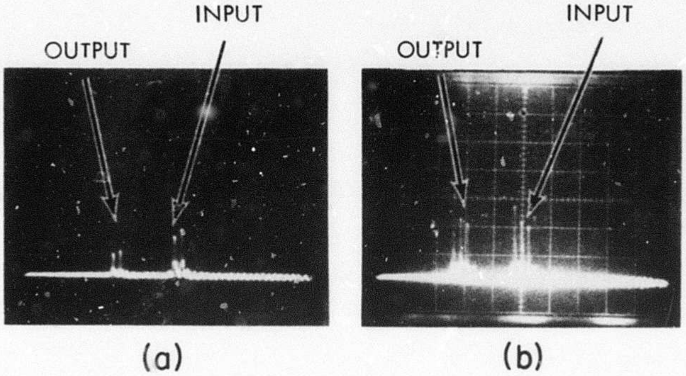

Fig. VI-13. Output and input of AHN experiment. Total pressure: 200 Torr. Gas mixture: $CO_2:N_2:He = X:4:100$. Intensity of input pulse: 0.75 (MW/cm^2)/div. Intensity of output pulse: 1.12 (MW/cm^2)/div. Time: 50 ns/div. (a) Amplifier off. (b) Amplifier on.

the oscilloscope display for this case.

In order to probe the temporal evolution of the three-electrode laser amplifier, we first fired the amplifier and, after a chosen delay time, the oscillator. In general, the oscillator was fired ~30 μs after the onset of the discharge in the amplifier for two reasons: first, we wished to avoid the effects on our measurements of the shock waves generated by the discharge. Second, we wished to be certain that the symmetric stretching (SS) and bending (B) modes of CO_2 had equilibrated with each other at slightly above the kinetic temperature of the gas.[2]

Measurements on the amplification of high-intensity ns pulses were made at 200 Torr of $CO_2:N_2:He$ mixtures. The ratio of these mixtures was $CO_2:N_2:He = X:4:100$, where CO_2 partial pressure was varied from 3.5 to 35 Torr partial pressure. Small-signal gain of this three-electrode laser amplifier in each case was measured by a cw CO_2 laser.

Results

Figure VI-13b illustrates the input and output pulses when the amplifier is turned on. Four such measurements were made and their average was taken at each CO_2 partial pressure studied. We have found that the RG-8 cable used for the time delay introduces some distortion in the input signals. This distortion can be accounted for if the first (second) pulses of the input and output pulses from the amplification measurements are compared with the first (second) pulses of the input and output pulses when the amplifier was evacuated.

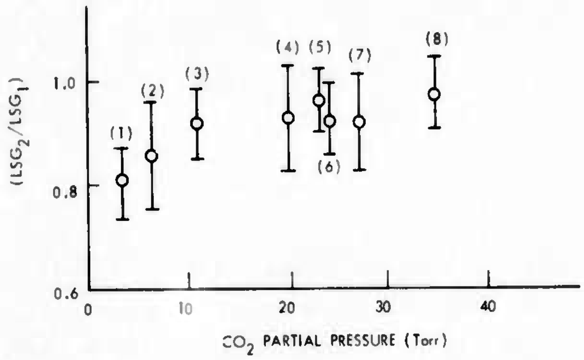

Fig. VI-14. LSG_2/LSG_1 vs partial CO_2 pressure.

Table VI-1. Experimental results.

No.	P_{CO_2} (Torr)	SSG ±10%	LSG_1 ±10%	T^a_{vv} (°K)	$\frac{\Delta N_u}{N_u(0)}$
1	3.5	0.50	0.30	1180	0.12
2	6.5	0.70	0.36	1150	0.11
3	11.0	1.50	0.76	1155	0.14
4	20.0	1.00*	0.52	845	0.11
5	23.0	1.50*	0.75	860	0.13
6	24.0	2.00*	0.96	890	0.14
7	27.0	2.50*	1.15	935	0.15
8	35.0	2.00	0.81	830	0.10

Notes: *Not measured directly; calculated from LSG_1 and the peak intensity of the pulse.

$SSG = \frac{I_{in} - I_{out}}{I_{in}}$ is the small-signal gain across the tube. Here the intensity is less than 1 W/cm^2.

LSG_1, large-signal gain of the first pulse.

T^a_{vv}, temperature of the asymmetric stretching mode calculated from SSG.

$\frac{\Delta N_u}{N_u(0)}$, fractional depletion of the 00°1 population by an ns pulse.

ΔN_u, calculated from the large-signal gain and the intensity of the pulse.

$N_u(0)$, obtained from SSG.

Figure VI-14 shows LSG_2/LSG_1 vs the partial pressure of CO_2 studied. We have

$$LSG_i \triangleq \frac{I_{out\,i} - I_{in\,i}}{I_{in\,i}},$$

where i = 1, 2, with 1 and 2 representing first and second pulses. LSG_1 (LSG_2) is the large-signal gain of the first (second) pulses. The following observations can be made from these measurements. (i) LSG_2/LSG_1 is less than unity. This is to be expected, since the first pulse had depleted a fraction of the population from the 00°1 level of CO_2. (ii) The ratio LSG_2/LSG_1 is approximately 0.8 at CO_2 partial pressure of 3.5 Torr and the ratio increases slowly to 0.9 as CO_2 partial pressure increases to 20 Torr or higher, which is as expected, since the 00°1 level of CO_2 was being repopulated by the higher $0n^{\ell}m$ levels at a rate[3] that is directly proportional to the CO_2 partial pressure.

Table VI-1 gives other experimental results of interest. We found that the large-signal gain of our pulses is approximately one-half the small-signal gain, and the fractional depletion of the 00°1 level, $\frac{\Delta N_u}{N_u(0)}$, is between 0.10 and 0.15.

A theoretical model for the amplification of high-intensity nanosecond pulses is being developed. We shall present the theory, and make a comparison of theory and experiment in a future report.

References

1. Y. Manichaikul, "Generation and Amplification of High-Intensity, Nanosecond TEA CO_2 Laser Pulses," Quarterly Progress Report No. 110, Research Laboratory of Electronics, M.I.T., July 15, 1973, pp. 118-121.
2. D. L. Lyon, IEEE J. Quant. Electronics, Vol. QE-9, No. 1, pp. 139-153, January 1973.
3. I. Burak, Y. Noter, and A. Szöke, IEEE J. Quant. Electronics, Vol. QE-9, No. 5, pp. 541-544, May 1973.

A THEORY OF FORCED MODE LOCKING*

by H. A. Haus**

Abstract:

A new theory of forced mode locking is developed in the frequency domain. Replacement of the discrete cavity mode spectrum by a continuum reduces the problem to the solution of an eigenvalue problem of a total differential equation. We treat the mode locking of both a homogeneous and an inhomogeneous laser medium. The equations of mode locking by modulation of resistance and reactance in a cavity containing a homogeneously broadened laser medium are those of the harmonic oscillator of quantum mechanics. The "supermodes" of the mode locked system as functions of the continuum frequency are the eigenfunctions of the harmonic oscillator. A stability analysis of this system shows that only the lowest order supermode is stable with respect to small perturbations. The properties of this supermode are the same as those previously derived by Kuizenga and Siegman. The same analysis lends itself to the treatment of mode locking of a cavity containing an inhomoge-

* Work supported by Joint Services Electronics Program (Contract DAAB07-71-C-0300). US Army Research Office - Durham (Contract DAHC04-72-C-0044).

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neously broadened laser medium. Again one finds that the lowest order supermode with a single peak (as a function of frequency) is stable, all higher order modes are unstable. The transient build-up of a supermode is analyzed. Effects of modulator detuning are treated.

Introduction

The literature on mode locking is large, comprising both theoretical and experimental investigations[1]. The theoretical investigations may be roughly divided into two groups, (a) those which treat the mode locking process in the time domain, e.g. the excellent paper by Kuizenga and Siegman[2], and (b) those which treat the phenomenon in the frequency domain (notably the paper by McDuff and Harris).[3] In this paper we present a useful extension of the mode locking analysis in the frequency domain. When setting up the analysis of saturable absorber mode locking by a somewhat novel approach we soon discovered that this approach lent itself very well to a better understanding of the forced mode locking phenomenon and permitted a broader investigation of the influence on mode locking of changes in various parameters.

This paper is devoted to forced mode locking. In particular the following issues are addressed: 1. Mode locking of a homogeneously broadened laser via single side-band-pair generation, using both amplitude and phase modulation. 2. Tuning of the mode locking element drive away from the frequency corresponding to the cavity mode spacing. 3. Changes in position of the mode locking element along the cavity. 4. The difference of mode locking between lasers possessing inhomogeneous versus homogeneous broadening. 5. Transient build-up of a mode locked

pulse. 6. Stability of the mode locked pulses.

We consider the problem in the frequency domain utilizing the theory of injection locking as recently described by Kurokawa.[4] Mode locking differs from injection locking only in that the injected signal is generated internally to the oscillator cavity by the signal present within the cavity. A complication of the laser mode locking is that, if it is to be of any use, many cavity modes must be locked and hence the analysis calls for the solution of many simultaneous equations. This fact, however, can be bent to one's advantage. As long as the modes under consideration are numerous, one may treat them as a continuum and replace the difference equation by a differential equation. This approximation has the advantage that the eigenvalue techniques of differential equations can be brought to bear on the mode locking problem. The approximation of replacing a discrete set of modes by a continuum implies that the pulses predicted by this theory must be separated (in time) sufficiently so as not to overlap. As soon as overlap of the pulses becomes important, the replacement of the discrete spectrum by a continuum is not legitimate.

II. Mode Locking as Injection Locking

Injection locking of a cavity with a single mode containing a negative resistance may be analyzed with the aid of the equivalent circuit of Fig. 1. Here the impedance $-Z_a(A)$ represents the active impedance which is in general complex and a function of the amplitude of the complex circulating current I. If a voltage source E is inserted between the two impedances as indicated, then the equation for this injection locked oscillator is given by

$$E = [Z_c(\omega) - Z_a(A)] I \tag{2.1}$$

In the sequel we shall follow Kurokawa's notation who uses I for the complex current amplitude and A for its (real) magnitude.

Equation (2.1) can be adapted to the case of a mode locked cavity involving the interaction among many axial modes. We comprise the current amplitudes of all the modes of interest in a current amplitude column vector I. If the modes in the cavity are uncoupled, in the absence of a mode locking crystal, then the impedance matrix of both the passive and the active part of the circuit is diagonal. We shall simplify the analysis by assuming that the cavity impedance may be described by the diagonal matrix

$$R_c \, [\, \underset{\sim}{1} + j\, \underset{\sim}{x}_c(\omega_i)] \tag{2.2}$$

where R_c is the equivalent cavity series resistance assumed identical for all modes; $\underset{\sim}{x}_c$ contains the normalized reactive components of the impedances of each mode, a diagonal matrix;

to the extent that the reactances may differ from mode to mode, $\underset{\sim}{x}_c$ is not proportional to the identity matrix. We indicate by ω_i the frequency of the injected signal (as well as oscillation) for a general cavity mode. The frequencies of the driven oscillations of all other modes are "spaced" from the center frequency by a multiple of ω_m, where ω_m is the frequency of modulation. (See Fig. 2)

Mode locking produces sidebands for each of the oscillating amplitudes. The generalization of (2.1) to this case is

$$\underset{\sim}{M}\,\underset{\sim}{I} = [\,\underset{\sim}{1} - \underset{\sim}{z}_a(A) + j\,\underset{\sim}{x}_c(\omega_i)\,]\,\underset{\sim}{I} \qquad (2.3)$$

$\underset{\sim}{z}_a(A)$ is the normalized impedance matrix contributed by the active medium, a diagonal matrix, but not proportional to the identity matrix. On the left hand side appears the matrix $\underset{\sim}{M}$ which is a nondiagonal matrix and accounts for the generation of sidebands by mode locking. The detailed form of the matrix $\underset{\sim}{M}$ depends upon the nature of the mode locking scheme.

It should be pointed out that, in accordance with Kurokawa, we use series equivalent circuits to represent the cavity modes interacting with the laser medium. If E-field amplitudes of the laser modes were identified with voltages and H-field amplitudes with currents, a parallel equivalent circuit would result. The reader has then the choice of either (a) adhering to this identification and reading "admittance" instead of "impedance" everywhere in the sequel, or to identify currents with E-field amplitudes and voltages with H-field amplitudes.

III. Modulation of Resistance.

Suppose the cavity contains a resistive medium which is modulated in time, so that its resistivity has the time dependence:

$$\text{constant} \; + (1 - \cos \omega_m t)$$

If the modulated medium is positioned inside the cavity so that all modes have the same field distribution through the medium (i.e. the mode locking element is relatively thin and is near one of the mirrors) then the medium contributes to the resistance R_c of the cavity a time dependent component

$$R_m (1 - \cos \omega_m t)$$

where the remaining time independent contributions may be lumped with R_c.

A current $|I_k| e^{j\omega_k t}$ pertaining to the k-th mode, flowing through R_m, produces a voltage

$$- R_m [1 - \cos \omega_m t] |I_k| e^{j\omega_k t} \tag{3.1}$$

$$= - R_m |I_k| [e^{j\omega_k t} - \tfrac{1}{2} e^{j(\omega_k + \omega_m)t} - \tfrac{1}{2} e^{j(\omega_k - \omega_m)t}]$$

The (equivalent injection locking) voltage at the frequency

of the k-th mode is therefore:

$$E_k = R_m \left[\tfrac{1}{2} I_{k-1} - I_k + \tfrac{1}{2} I_{k+1}\right] \qquad (3.2)$$

The modulation matrix of (2.3) becomes, in the present case:

$$\underset{\sim}{M} = \frac{R_m}{2R_c} \begin{bmatrix} 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \end{bmatrix} \qquad (3.3)$$

Introducing (3.3) into (2.3) one obtains a second order difference equation.

We may now make a series of approximations so as to enable us to find simple solutions to this equation. First of all, if many modes are to be locked, and if the variation of the gain over a frequency range corresponding to the mode spacing is small, then the difference equation can be replaced by a second order differential equation. One has

$$\frac{R_m}{2R_c} \left[I_{k+1} - 2I_k + I_{k-1}\right] \rightarrow \frac{R_m}{2R_c}\, \omega_m{}^2 \, \frac{d^2 I}{d\omega^2} \qquad (3.4)$$

where I, the column vector containing the (complex) current amplitudes of the modes, is now replaced by a continuous function

of frequency deviation from material line-center, ω, and ω_m is the frequency separation of the spectral lines. The modes are now identified by the continous variable $\omega = \lim_{\substack{k \to \infty \\ \omega_m \to 0}} k\,\omega_m$, where k is the mode number and assumes both positive and negative values. Defining the modulation coefficient $M = \frac{R_m}{2R_c}$ one obtains for (2.3)[5]:

$$M\,\omega_m^2 \frac{d^2 I}{d\omega^2} = [1 - z_a(A, \omega) + j\,x_c(\omega)]\; I\,(\omega) \qquad (3.5)$$

In the sequel we specialize to specific cases.

IV. Homogeneously Broadened Laser Linewidth.

When the laser is homogeneously broadened, then the impedance contributed by the active medium z_a, is proportional to $1/(1 + P/P_s)$ where P is the total power in the laser cavity in all modes and P_s is the saturation power. This is true over the entire linewidth, i.e. for all values of ω of the function $z_a(A, \omega)$. We shall make the further approximation that the gain profile (lineshape) of the laser medium, and hence the frequency dependence of negative resistance r_a can be replaced by a parabola. This assumption holds provided the loss line is reasonably close to the peak of the gain curve so that no modes oscillate within the frequency range in which the line shape of the medium differs appreciably from the parabolic approximation. With these assumptions we have

$$z_a(A, \omega) = r\left\{1 - \frac{\omega^2}{\omega_M^2} - j\,\frac{\omega}{\omega_M}\right\} \tag{4.1}$$

where ω_M^2 is related to the curvature at the peak of the gain line and is a measure of the laser medium bandwidth. The dependence on power P of r is

$$r = \frac{r_o}{1 + \dfrac{P}{P_s}} \tag{4.2}$$

where r_o is the small signal value of r.

We have added a reactive contribution to the laser medium which, by the Kramers-Kronig relations, must be associated with a frequency dependent real part.

Introducing (4.1) into (3.5) we obtain

$$M\,\omega_m^{\,2}\,\frac{d^2 I}{d\omega^2} = \left[1 - r\left(1 - \frac{\omega^2}{\omega_M^{\,2}}\right) + rj\,\frac{\omega}{\omega_M} + j\,x_c(\omega)\right] I \qquad (4.3)$$

In (4.2) the assumption was made that the gain of the medium does not get modulated by the passing laser pulses; the medium gain is pulled down on a time average basis only. This assumption amounts to the statement that the population recovery time (T_1) of the medium is long so that the gain change is caused by the depleting action of many successive pulses, each pulse changing the population slightly. Gain modulation does, in fact, tend to counteract mode locking and will be treated in another publication. Hence the theory developed in this paper, like the theory of Siegman and Kuizenga, is good only for relatively long T_1.

Next consider the meaning of $x_c(\omega)$. This is the cavity mode reactance as a function of mode number, here indicated by the continous variable ω. x_c is zero if the mode is excited at resonance and is nonzero for a frequency deviation $\delta\omega$ from mode resonance and given by

$$x_c = 2\,\frac{\delta\omega}{\omega_0}\,Q \qquad (4.4)$$

where Q is the quality factor of the cavity mode and ω_o is the resonance frequency (which may be considered to be a constant for all modes within the narrow laser line). Eq. (4.3) is particularly simple, if we require that

$$x_c(\omega) + r\frac{\omega}{\omega_M} = 0 \tag{4.5}$$

This implies that

$$\frac{\delta\omega}{\omega_o} = -\frac{r\omega}{\omega_M}\frac{1}{2Q}$$

namely that the frequency deviation $\delta\omega$ from mode resonance is proportional to the frequency separation from the material line center. Such a condition can be achieved by proper adjustment of the frequency of the modulator. Indeed, we may write $\omega = k\,\omega_m$ where n is the order of the mode as measured from material line center. Therefore, if the modulator frequency ω_m is set so that

$$\delta\omega = k(\omega_m - \Delta\omega) = -r\frac{k\omega_m}{\omega_M}\omega_o\frac{1}{2Q}$$

we find for ω_m:

$$\omega_m = \Delta\omega\left[1 + r\,\omega_m\frac{\omega_o}{\omega_M}\frac{1}{2Q}\right]^{-1} \tag{4.6}$$

In this way, we have achieved the desired frequency deviation as a function of mode number k; (4.6) implies that the modulation frequency is set equal to the mode resonance frequency separation as modified by the laser medium. With (4.5) satisfied, Eq. (4.3) has a pure real coefficient on the right hand side. The resulting equation is identical with the harmonic oscillator equation of quantum mechanics, where I plays the role of the wave function and ω the role of the spacial variable.

We can now bring to bear the entire formalism of the quantum mechanical harmonic oscillator on the problem of loss mode locking in a homogeneously broadened medium. The solutions of (4.3) are the well known Hermite polynomials [6]

$$\psi(\xi) = H_n(\xi) \exp -\tfrac{1}{2} \xi^2$$

where the variable ξ is a spacial variable normalized in terms of the parameters of the harmonic oscillator.

$$\xi = \sqrt[4]{\frac{m\,k}{\hbar^2}}\; x$$

The role of the harmonic oscillator quantity $\hbar^2/2m$ is played by the parameter $M\omega_m{}^2$ and the spring constant has to be identified with $2r/\omega_M{}^2$. With this identification we have

$$I(\Omega) = H_n(\Omega) \exp - 1/2\, \Omega^2 \qquad (4.7)$$

where

$$\Omega = \frac{\omega}{\sqrt[4]{\frac{M}{r}}\,\sqrt{\omega_m\,\omega_M}} \tag{4.8}$$

The eigenvalues of energy of the harmonic oscillator follow the well known law $h\nu(n + 1/2)$ where n is an integer. The role of energy is played by the parameter $r - 1$ in equation 4.3. We find for the quantization of this parameter

$$r - 1 = \frac{2\sqrt{Mr}\;\omega_m\,(n + 1/2)}{\omega_M} \tag{4.9}$$

Equation (4.7) defines "supermodes" or "mode locking modes" of the mode locked oscillator. The higher the order n of the "supermode", the more structure in the frequency spectrum. Since the Fourier transform of the spectrum (4.7) leads to the same time dependence, we find that mode locked solutions may exist of higher and higher temporal structure exhibiting more and more pulses. (See Fig. 3)

For a given excess gain, $r_o > 1$, and given modulation M, the bandwidth of the lowest order supermode $(n = 0)$ is proportional to $\sqrt{\omega_M}$, i.e. the square root of the pressure in a homogeneously broadened gas laser. Equation (4.9) determines the power in the mode locked cavity for the various "supermodes". Indeed by solving for r from (4.9) and then using (4.2) we find the following relationship for the power

$$\frac{P}{P_s} = \frac{r_o}{1 + \mu + \sqrt{2\mu + \mu^2}} - 1 \tag{4.10}$$

where $\mu = \frac{1}{2}(2n + 1)^2 M \left(\frac{\omega_m}{\omega_M}\right)^2$

In the limit of a small modulation coefficient M, (4.10) becomes

$$\frac{P}{P_s} = r_o - 1 - 2 r_o\sqrt{M} \frac{\omega_m}{\omega_M} (n + 1/2) \qquad (4.10a)$$

We see that, for a given modulation coefficient, the power decreases with "supermode" order. Another interesting fact may be noted from (4.9). The highest order supermode supportable in a given cavity of specified small signal gain, is obtained by setting $r = r_o$ in (4.9). We note that the smaller M, the modulation coefficient, the higher the allowed order. This appears somewhat counter-intuitive until one realizes that we have not discussed which supermodes are stable and thus would, in fact, exist in the steady state. We have merely found that weak modulation leads to weak selectivity among the supermodes and hence to presumably a noisy output. The larger the modulation coefficient M, the fewer supermodes are allowed. The question of stability of the supermodes will be taken up in Section XI.

The bandwidth is given roughly by the turning points of the Hermite solutions, namely by $2\omega_e$ where

$$r - 1 = r \frac{\omega_c^2}{\omega_M^2}$$

or, using (4.8) and (4.10)

$$2\,\omega_e = 2\,\omega_M \sqrt{2\,\sqrt{M/r}\;\frac{\omega_m}{\omega_M}\;(n + 1/2)} \tag{4.11}$$

The bandwidth is proportional to the 4-th root of the modulation coefficient and the square root of $n + 1/2$. Our analysis is applicable, as long as $(2\omega_e/\omega_m) >> 1$, or

$$2\sqrt{2\,\sqrt{M/r}\;\frac{\omega_M}{\omega_m}\;(n + 1/2)} >> 1 \tag{4.12}$$

which is not too severe a restriction as long as $\omega_M/\omega_m >> 1$.

V. Phase Modulation

Next, we investigate mode locking via modulation of a reactance. Suppose the reactance X_m is modulated so that it has a time dependence

$$X_m \, [\, 1 - \cos \omega_m t] \tag{5.1}$$

If one defines the modulation coefficient

$$M \equiv \frac{X_m}{2R_c} \tag{5.2}$$

one obtains an equation identical to (4.2) except that now M on the left hand side of (4.2) has to be multiplied by $-j$.

$$jM\,\omega_m{}^2 \frac{d^2 I}{d\omega^2} = \left[1 - r\left(1 - \frac{\omega^2}{\omega_M{}^2}\right) + rj\,\frac{\omega}{\omega_M} + jx_c(\omega) \right] I \tag{5.3}$$

This equation can be reduced to the form of (4.2) through a change of variables. One defines the new independent variable

$$\omega'^2 = \omega^2/\sqrt{j} \tag{5.4}$$

With this change of variable, one finds

$$M\,\omega_m{}^2 \frac{d^2 I}{d\omega^2} = \left\{ \left[\frac{1 - r}{\sqrt{j}} + \sqrt{j}\left(x_c + r\,\frac{\omega}{\omega_M}\right)\right] + r\,\frac{\omega'^2}{\omega_M{}^2} \right\} I \tag{5.5}$$

Clearly the solutions of this equation are the same as before,

$$I(\omega) = H_n \left[\frac{1}{\sqrt{2}} (1 - j)\Omega\right] \exp - \frac{1}{2\sqrt{2}} (1 - j)\Omega^2 \qquad (5.6)$$

where the normalized frequency Ω has been defined in (4.8). The phase of the mode locked complex current amplitudes varies across the frequency profile. Using the fact that the eigenvalues of (5.5) are the same as the ones previously obtained in (4.9) one has

$$\frac{1}{\sqrt{2}}\left[(r - 1)(1 - j) - (1 + j)\left(x_c + r \frac{\omega}{\omega_M}\right)\right] = \frac{2\sqrt{Mr}\ \omega_m (n + 1/2)}{\omega_M} \qquad (5.7)$$

Equating the imaginary parts of this equation, one finds the amount of the detuning of the cavity modes from resonance.

$$(r - 1) = -\left(x_c + r \frac{\omega}{\omega_M}\right)$$

We find that the modulator frequency ω_m obeys a slightly different condition than (4.6), if solution (5.6) is to apply. Indeed, using (4.4) we find

$$x_c = 2 \frac{\delta\omega}{\omega_o} Q = 2 \frac{Q}{\omega_o}\left[\delta\omega_o + k(\omega_m - \Delta\omega)\right] = -r \frac{k\omega_m}{\omega_M} - (r - 1) \qquad (5.8)$$

where $\delta\omega_o$ is the frequency deviation of the oscillation from the resonance frequency of the mode at laser medium line center $(k = 0)$. To the extent that $\Delta\omega - \omega_m \neq 0$, this frequency deviation changes as one goes away from line center. Matching the k-dependent parts of (5.8), one has:

$$\omega_m = \Delta\omega \left[1 + r \frac{\omega_o}{\omega_M} \frac{1}{2Q} \right]^{-1}$$

which is identical to (4.6). Hence, solution (5.6) requires that the modulation frequency be tuned to the mode separation frequency, as modified by the reactance of the laser medium. Matching the k-independent parts of (5.8) one has:

$$\delta\omega_o = - (r - 1) \frac{\omega_o}{2Q}$$

The oscillations occur off mode resonance (as modified by the laser medium). Equating the real parts of (5.7), one finds a relation for the power of oscillation.

$$\left(\frac{r_o}{1 + \frac{P}{P_s}} - 1 \right) = \frac{\sqrt{2}\sqrt{Mr_o}\,\omega_m (n + 1/2)}{\sqrt{1 + \frac{P}{P_s}}\,\omega_M}$$

We have found that phase modulation works similarly to amplitude modulation, but that the cavity modes do not oscillate at cavity line-center.

A comparison of the pulses produced by amplitude modulation, or phase modulation, can be made if one studies the inverse Fourier transform of the spectra. We concentrate solely on the lowest order supermode. Taking into account that

$$\int_{-\infty}^{\infty} d\omega\, e^{j\omega t}\, e^{-(a-jb)\omega^2/2}$$

$$= \sqrt{\frac{\pi}{a+jb}}\; e^{-\frac{(a+jb)t^2}{(a^2+b^2)2}}$$

we may make the following identifications:

a) AM modulation

$$a = \frac{1}{\sqrt{\frac{M}{r}}\,\omega_m\ \omega_M}$$

$$b = 0$$

so that the exponential multiplied by the carrier becomes

$$\exp -\left(\frac{1}{2}\sqrt{M/r}\ \omega_m\ \omega_M t^2\right) \exp j\omega_o t$$

The pulsewidth, defined as the width between the halfpower points is

$$\tau = 2\sqrt{a\ \ln 2} = 2\sqrt{\ln 2\,\frac{1}{\sqrt{\frac{M}{r}}\,\omega_m\,\omega_M}}$$

b) FM modulation

$$a = \frac{1}{\sqrt{2}} \; \frac{1}{\sqrt{\frac{M}{r}}\, \omega_m \, \omega_M} = b$$

The exponential is

$$\exp - \frac{1}{2} \sqrt{M/2r}\, \omega_m \, t^2 \; \exp j[\omega_o t - \frac{1}{2} \sqrt{M/2r}\, \omega_m \, \omega_M \, t^2]$$

The width of the phase mode locked pulse is by $\sqrt{2}$ greater than that of the amplitude mode locked pulse. This is in agreement with a result due to **Kuizenga and Siegman** obtained by a different method, if their notation is brought into correspondence with our notation.

In addition, the frequency of the phase-modulation pulse is chirped, having the "instantaneous" frequency

$$-\sqrt{M/2r}\, \omega_m \, \omega_M \, t$$

The frequency decreases with time for the assumed modulation of reactance (5.1).

VI. Modulator Detuning, Resistance Modulation

We shall now suppose that the modulator frequency ω_m is not set equal to the frequency spacing of the cavity modes (as determined in the presence of the laser medium). Therefore, the modes at greater "distance" from line center, become progressively more detuned. The reactance x_c of a cavity mode at a deviation $\delta\omega$ from a center frequency ω_o is given by (4.4). If ω_m is not set equal to the separation

$$\Delta\omega \left[1 + r \frac{\omega_o}{\omega_M} \frac{1}{2Q}\right]^{-1} \tag{6.1}$$

of the cavity resonance frequencies in the presence of the laser medium, the function in the brackets on the right hand side of (4.3), i.e. the impedance of modes and laser medium as a function of mode number k, becomes complex. One has

$$x_c + r \frac{\omega}{\omega_M} = \frac{2Q}{\omega_o} k(\omega_m - \Delta\omega) + r \frac{k\omega_m}{\omega_M} \tag{6.2}$$

If one defines the detuning parameter δ by

$$\delta \equiv \left[\omega_m \left(1 + r \frac{\Delta\omega_c}{2\omega_M}\right) - \Delta\omega\right] / \Delta\omega_c \tag{6.3}$$

where $\Delta\omega_c$ is the cavity mode bandwidth one obtains:

$$x_c(\omega) + r \frac{\omega}{\omega_M} = 2\,\delta\,\frac{\omega}{\omega_m} \tag{6.4}$$

Introducing this dependence into (4.3), we find the differential equation

$$M\omega_m{}^2 \frac{d^2 I}{d\omega^2} = \left[1 - r\left(1 - \frac{\omega^2}{\omega_M{}^2}\right) + 2j\,\delta\,\frac{\omega}{\omega_m}\right] I \tag{6.5}$$

Again it is not difficult to find a solution for this differential equation using a shift in origin of the independent variable. Introducing the shifted frequency ω' by the relation

$$\omega' \equiv \omega + j\,\delta\,\frac{\omega_M{}^2}{r\,\omega_m}$$

we may obtain the new differential equation

$$M\,\omega_m{}^2 \frac{d^2 I}{d\omega^2} = \left[1 - r\left(1 - \frac{\omega'^2}{\omega_M{}^2}\right) + \frac{\delta^2 \omega_M{}^2}{r\,\omega_m{}^2}\right] I \tag{6.6}$$

Again, this is the harmonic oscillator equation. Note, however, that the shift of origin of the independent variable is imaginary. The lowest order supermode can be shown to be time delayed (with respect to the modulator phase) by the amount:

$$\tau_{delay} = \delta \frac{\omega_M}{\omega_m} \frac{1}{\sqrt{Mr}} \frac{1}{\omega_m} \tag{6.7}$$

Writing down the eigenvalue relation for this equation one obtains

$$\frac{r_0}{1 + \frac{P}{P_s}} - 1 - \frac{\delta^2 \omega_M{}^2}{r\,\omega_m{}^2} = \frac{2\sqrt{Mr_0}\,\omega_m (n + 1/2)}{\sqrt{1 + \frac{P}{P_s}}\;\omega_M} \tag{6.8}$$

One finds that mode locking may be prevented by excessive detuning of the modulation frequency. The detuning introduces a term analogous to a loss increase of the cavity.

The effect of detuning may be interpreted as follows: Minimum loss of pulse would be achieved if the pulse passed the mode locking crystal at the time of its peak transmission. The repetition rate, however, is inconsistent with the cavity resonance frequency spacing. Hence, the cavity reactances are nonzero. This calls for a reactive contribution of the mode locking crystal which is achieved when the peak of the pulse passes at a time different from the peak transmission time. Additional loss due to the modelocking crystal is then unavoidable.

VII. Changes in Position of Mode Locking Crystal

So far we have assumed that the crystal was near one of the mirrors so that all cavity modes had roughly the same field through the crystal. The equivalent circuit for a cavity containing a resistivity ρ and having normalized magnetic field patterns of modes k and ℓ, $\bar{h}_k$ and $\bar{h}_\ell$ respectively, possesses a coupling resistance $R_{k\ell}$ between the modes k and ℓ of the form

$$R_{k\ell} \propto \frac{\int \rho\, \overline{h_k^*} \cdot \overline{h_\ell}\, dv}{\int \overline{h_k^*} \cdot \overline{h_k}\, dv} \tag{7.1}$$

where the integrals are taken over the cavity volume.

Now consider an equivalent current $I_k \exp(j\omega_k t)$ of the k-th mode and ask for the voltage at the frequency $k + 1$. This is given by the coupling resistance $R_{k+1,\,k}$ which has the following dependence upon the field patterns

$$R_{k+1,\,k}(t)\, I_k\, e^{j\omega_k t} \quad \frac{\int \rho\,(t)\, \overline{h_{k+1}^*} \cdot \overline{h_k}\, dv}{\int \overline{h_{k+1}^*} \cdot \overline{h_{k+1}}\, dv}\, I_k\, e^{j\omega_k t} \tag{7.2}$$

Suppose that we have standing waves inside the cavity and all modes of interest have roughly the same cross-sectional depen-

dence. In this case, one finds for the integral in the limit of a crystal of short length L, positioned at a distance x_o from one of the mirrors.

$$R_{k+1,\,k} \propto \frac{\lim\limits_{\Delta\omega\, L/c \to 0} \int_{x_o}^{x_o + L} dx \cos \beta_k x \cos \beta_{k+1} x}{\int_{x_o}^{x_o + L} dx \cos^2 \beta_{k+1} x} \tag{7.3}$$

$$= \cos \frac{\Delta\omega}{c} x_o \qquad \text{where} \quad \beta_k = \frac{2\pi}{\lambda_k} = (\omega_o + k\Delta\omega)/c$$

We thus find that the strongest coupling is achieved when the position of the crystal is near one of the mirrors and no coupling is achieved if the crystal is positioned at the center of the cavity. This finding depends in part on the assumption that the mode locking crystal produces only a single pair of sidebands.

VIII. Higher Order Sideband Generation

The generalization to include the case in which more than a single pair of sidebands is generated is not difficult; what is difficult is the solution of the resulting equation. Indeed if, say, two pairs of sidebands are generated by the modulation, located symmetrically around the modulated carrier, then the second derivative of the current amplitude has to be replaced by a superposition of a fourth derivative and a second derivative. The resulting equation is much more difficult than the one obtained thus far. Whereas solutions for equations of this form have been obtained in the literature[7], they are complicated and have not yet been explored with the present purpose in mind.

Another way of generalizing the equations is to replace $M \frac{d^2}{d\omega^2}$ by an operator. An integral operator is found to be appropriate for the analysis of a saturable absorber and will be taken up elsewhere.

IX. Inhomogeneously Broadened Medium

If the medium is inhomogeneously broadened, then the power in any particular cavity mode determines the pulldown of the gain line at that particular cavity frequency (provided, of course, that the separation of the cavity modes is large compared with the homogeneous linewidth of the medium). The adaptation of (4.3) to the case of an inhomogeneously broadened medium is accomplished by changing the character of the dependence of the negative resistance upon intensity. For simplicity we omit the reactive contribution of the laser medium, an assumption justified by the fact that the overall gain profile tends to get "flattened" by the simultaneous oscillations of many cavity modes and the individual modes oscillate at the center of the local hole. The equation is:

$$M\ \omega_m^2 \frac{d^2 I}{d\omega^2} = \left[1 - \frac{r_o\left(1 - \frac{\omega^2}{\omega_M^2}\right)}{1 + \frac{A^2}{A_s^2}} + j\ x_c(\omega)\right] I \qquad (9.1)$$

where A in the denominator is the square of the amplitude at the particular frequency of interest and A_s^2 is proportional to the saturation intensity. (9.1) is a nonlinear differential equation for the current I (remember that A is the magnitude of the current). It is a difficult equation to solve; yet it is possible to gain considerable insight into the nature of its solutions by giving it a physical interpretation. Let us separate the complex current amplitude into an amplitude and a phase factor

$$I = A\, e^{j\phi} \tag{9.2}$$

Taking the second derivative of (9.2) one obtains

$$\frac{d^2 I}{d\omega^2} = \ddot{A}\, e^{j\phi} + 2j\dot{\phi}\, \dot{A}\, e^{j\phi} + j\ddot{\phi}\, A\, e^{j\phi} - \dot{\phi}^2\, A\, e^{j\phi} \tag{9.3}$$

Introducing (9.2) and (9.3) into (9.1) one has

$$M\, \omega_m{}^2 \quad (\ddot{A} - \dot{\phi}^2 A) = \left\{ 1 - \frac{r_o \left[1 - \frac{\omega^2}{\omega_M{}^2} \right]}{1 + \frac{A^2}{A_s{}^2}} \right\} A \tag{9.4}$$

$$M\, \omega_m{}^2 \quad (A\ddot{\phi} + 2\dot{A}\, \dot{\phi}) = x_c A. \tag{9.5}$$

Here we have written $(\dot{\ })$ for $d/d\omega(\ )$ in order to emphasize the analogy with the equations of motion of a particle in polar coordinates; the frequency variable plays the role of "time", the amplitude A is analogous to "radius". The force field is "radius" and "time" dependent. If one assumes that all cavity modes oscillate at line center of the respective cavity resonances, then $x_c = 0$ everywhere and the force field becomes a central one. In this case, one may define a potential function, the derivative of which gives the force. The potential function plotted against "radius" A as a function of "time" ω is shown in Fig. 4.

Consider first the case in the absence of modulation. In the particle motion analog, this corresponds to a massless particle. The particle will always look for a position in which it is exposed to zero force field. As we can see from Fig. 4., the

potential hill travels as a function of time towards the origin and then the origin becomes a hill in its own right. If one looks for a mode locked "supermode", one looks for the motion of a particle that starts with zero velocity at a time $t = 0$ (zero slope at center frequency), and then approaches the origin ($A = 0$) where it comes to a full stop. The equation of motion for the massless particle is then very simple, it stays at the top of the hill starting infinitesimally to the left and then searches for the position in which the force is infinitesimally small yet directed from right to left. When the potential hill has made it to the origin, the particle has followed it there and then stays there for ever. This case can be solved very simply analytically by setting the righthand side of equation (9.4) = 0.

Consider next the case when there is modulation and when the particle has to be assigned a nonzero mass. Let us consider at first the case of very small mass (modulation). At first, when the force field is strong, one may still consider the particle to seek out positions in the potential field at which it is exposed to practically zero force. This means that the particle follows roughly the same motion as we have determined earlier by assuming zero mass for the particle, except when it approaches the origin. Near the origin, the force field approaches zero, but the particle has acquired a finite velocity and kinetic energy. In order to come to a full stop at the origin, it has to expend the kinetic energy by climbing up a hill. This

means that, before the particle comes to full rest, the origin must have already risen above its surroundings so that a particle, near the end of its motion, will have to climb up a hill. Translated into the language of the mode locked "mode" this means that the frequency bandwidth of the mode is wider than in the absence of modulation (remember in the language of the equivalent particle the origin of the radial coordinate develops into a hill after the massless particle has reached the origin). Further, it is clear that a particle with mass requires an initial force in order to be nudged towards the origin, which means the particle has to start slightly to the left of the potential hill at $t = 0$. Again, in the language of mode locking, this corresponds to a power of the mode near line center that is smaller than the power in the absence of mode locking. The model is useful also for determining what happens when the strength of mode locking is increased excessively. This corresponds to a very massive particle. The potential, as defined, does not involve the mode locking amplitude M and hence, in the equivalent language of the particle, the initial hill becomes smaller, the heavier the particle. A supermode, symmetric at line center, corresponds to an initial condition for the analog particle starting at rest to the left of the potential hill. A particle which starts initially at rest slightly to the left of the potential hill may be too massive to make it near the origin before the potential distribution has changed beyond recall, pre-

venting the particle from acquiring sufficient kinetic energy. The particle never makes it to the origin. No symmetric supermode exists for excessively high values of M.

There are also higher order steady state solutions, if the modulation is not too strong. One may start the particle on the down slope of the hill. (Note the peak intensity at center frequency for this mode is less than for the lowest order supermode). The particle may go through the origin and climb up the hill on the other side. After oscillating back and forth, the particle can be brought to rest at the origin. Such solutions exist only for the particles of finite mass (i.e. finite modulation). Equation (9.4) has two adjustable apramerers, $r_0 - 1$, the excess small signal gain at line center $(\omega = 0)$ and the normalized modulation parameter $\sqrt{\frac{M}{r_0}}\frac{\omega_m}{\omega_M}$. For a range of these two parameters, solutions are obtained. Fig. 5 shows plots of A/A_s vs normalized frequency ω/ω_M, for various choices of the parameters $\sqrt{\frac{M}{r_0}}\frac{\omega_m}{\omega_M}$ and $r_0 - 1$. Fig. 6 shows the range in the $(r_0 - 1)$, $\sqrt{\frac{M}{r_0}}\frac{\omega_m}{\omega_M}$ plane over which mode locked solutions are obtained. Fig. 7 shows the decrease of the oscillation amplitude at line center $A(0)/A_s$, as a function of normalized modulation, for different excess small signal gains.

X. Transient Buildup of Mode Locked Laser

So far we have concentrated on obtaining steady-state supermode solutions. The case of the homogeneously broadened laser medium lends itself nicely for study of the buildup of the mode locked pulses (supermodes). Since we have treated the pulse spectrum as a continuum, and a continuous spectrum predicts one single pulse only, we must ask how the analysis can be adapted to the description of the buildup of a sequence of discrete pulses.

We must recall that the continuous spectrum was introduced in the first place in order to obtain an approximate solution to the difference equation. In fact, the supermodes obtained thus far, functions of frequency ω, must be interpreted as defining a closely spaced discrete spectrum, with spectral lines at $k\omega_m$, with k an integer. A perturbation of the frequency of each axial mode, i.e. replacement of $k\omega_m$ by $k\omega_m + \delta\omega_k$, implies a deviation of the oscillation frequency, Re $\delta\omega_k$, and growth of the axial mode at the rate $-\text{Im}\ \delta\omega_k$. A transient analysis then entails the introduction of the frequency perturbation $\delta\omega_k$ and an expansion of all parameters of the fundamental difference equation in terms of this frequency perturbation. One may go to the continuum limit, treating $\delta\omega$ as a function of $\lim k\omega_m \to \omega$, and carry out the expansion on the differential equation (4.3).

We shall disregard all derivatives of the parameters of (4.3) with respect to $\delta\omega$, except that of x_c. This is tantamount to disregarding all energy storages other than the electromagnetic energy storage of the cavity modes.

We now turn to the expansion of x_c, a function of mode

number k. If the oscillation of an axial mode occurs at a frequency different than the steady state frequency, then $x_c(k, k\omega_m)$ has to be replaced by $x_c = x_c(k, k\omega_m + \delta\omega_k) = x_c(k\omega_m) + \delta\omega_k \dfrac{\partial x_c(k, \omega)}{\partial\omega}$ or in the continuum limit

$$x_c = x_c(\omega) + \delta\omega\, x_c' \tag{10.1}$$

where $\delta\omega$ is itself a function of $\omega \to k\omega_m$ and we have set

$$x_c' \equiv \frac{\partial x_c(k, \omega)}{\partial\omega}$$

When the steady state oscillation frequencies are sufficiently near the resonance frequencies of the axial modes, x_c' can be treated as a constant.

$$M\omega_m{}^2 \frac{d^2 I}{d\omega^2} + \left[r\left(1 - \frac{\omega^2}{\omega_M{}^2}\right) - 1 - jx_c - rj\frac{\omega}{\omega_M}\right] I = j\,\delta\omega\, x_c'\, I \tag{10.2}$$

Clearly $x_c' > 0$ by Foster's reactance Theorem. Equation (10.1) can be solved simply if one assumes that the width of the medium $\sqrt{(r\omega_M{}^2)} \simeq \sqrt{(r_o\omega_M{}^2)}$, and hence the laser pulse width, is independent of time. We shall first assume that this is the case and then show the generalization to the case when the time variation of $r\,\omega_M{}^2$ is taken into account.

Suppose the modulation is resonant, $\omega_m = \Delta\omega\left(1 + r\dfrac{\omega_o}{\omega_M}\dfrac{1}{2Q}\right)^{-1}$, $x_c(\omega) + r\dfrac{\omega}{\omega_M} = 0$ in (10.1). If we set

$$r\left(1 - \frac{\omega^2}{\omega_M{}^2}\right) \cong \frac{r_o}{1 + \dfrac{P}{P_s}} - r_o\frac{\omega^2}{\omega_M{}^2} \tag{10.3}$$

and introduce this expression into (10.2) we obtain

$$M\omega_m^2 \frac{d^2 I}{d\omega^2} + \left[\frac{r_o}{1 + \frac{P}{P_s}} - 1 - \frac{r_o \omega^2}{\omega_M^2}\right] I = j\delta\omega\, x_c{}'\, I \qquad (10.4)$$

Setting

$$I = \frac{A\sqrt{T_R}}{\sqrt[4]{\pi\, \omega_p^2}} \exp - \frac{1}{2} \frac{\omega^2}{\omega_p^2} \equiv A\sqrt{T_R}\, v_o(\omega)$$

with (10.5)

$$\omega_p \equiv \sqrt[4]{M/r_o}\, \sqrt{\omega_m\, \omega_M}$$

where T_R is the time interval between successive pulses, we have $\int v_o^2(\omega)\, d\omega = 1$ and the energy in a single pulse is given by $A^2 T_R$, the power P is equal to A^2. Now $j\delta\omega A$ can be regarded as the time derivative of A, dA/dt, thinking of the amplitude of the Fourier spectrum to be a function of time.

Let us make this assertion mathematically precise. We have been treating the mode spectrum as a continuum. In fact, the mode spectrum is discrete, and the continuum replacement only served to obtain an approximate solution to the difference equation. Denote the Fourier transform of $v_o(\omega)$ by $v_o(t)$: Noting that we are dealing with a periodic process of long period T_R, we have for a sequence $v(t)$, of slowly varying amplitude $A(t)$:

$$\frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} dt\; e^{-jn\Delta\omega t}\; A(t)\; v_o(t)$$

$$\frac{1}{\sqrt{2\pi}}\, A(t) \int_{-T_R/2}^{T_R/2} dt\; e^{-jn\Delta\omega t}\, v_o(t)$$

$$= \frac{1}{\sqrt{2\pi}} \quad A(t) \int_{-\infty}^{\infty} dt \quad e^{-j\omega t} \, v_o(t)$$

$$= \frac{1}{\sqrt{2\pi}} \quad A(t) \; v_o \; (\omega)$$

The first step is legitimate, because $A(t)$ is assumed to vary negligibly within one period T_R and hence it can be pulled out from underneath the integration. The second step is justified by our assumption of negligible overlap of the pulses. The final expression shows that we may treat the Fourier spectrum both as time and frequency dependent.

Returning now to (10.4) we obtain, using (10.5):

$$\dot{A} = \frac{1}{x_c'} \left[\frac{r_o}{1 + \frac{A^2}{P_s}} - 1 - \sqrt{Mr_o} \; \frac{\omega_m}{\omega_M} \right] A \tag{10.6}$$

We may make the van der Pol assumption (really a necessary assumption in view of our disregard of the variation of $r\omega_M{}^2$) i.e.

$$\frac{1}{1 + \frac{A^2}{P_s}} \simeq 1 - \frac{A^2}{P_s}$$

With this assumption, (10.6) written in terms of $P \equiv A^2$ reduces to

$$\mathring{P} = 2\,\alpha_g P - 2\,\beta_g \frac{P^2}{P_s} \tag{10.7}$$

where

$$\alpha_g = \left[(r_o - 1) - \sqrt{M r_o}\,\frac{\omega_m}{\omega_M} \right] \frac{1}{x_c'}$$

and

$$\beta_g = r_o / x_c' \tag{10.8}$$

The solution to this equation is

$$\frac{P}{P_s} = \frac{e^{2\alpha_g t}}{1 + e^{2\alpha_g t}}\,\frac{\alpha_g}{\beta_g} \tag{10.9}$$

Figure 8 shows a plot of this solution vs normalized time $2\alpha_g t$, using the normalized power

$$P/\left(\frac{\alpha_g}{\beta_g}\,P_s\right)$$

We now take into account the change in the bandwidth of the laser medium by replacing Equation (10.3) with

$$r\left[1 - \frac{\omega^2}{\omega_M^2}\right] = \frac{r_o\left[1 - \frac{\omega^2}{\omega_M^2}\right]}{1 + \frac{A^2}{P_s}} \tag{10.10}$$

It should be noted that we assumed that the modulation frequency ω_m was equal to the cavity mode spacing, that the mode locking was on resonance. To the extent that ω_m is usually kept fixed, but the mode spacing varies with amplitude because of the changing dispersion of the laser medium this is an approximation. It can be shown, however, that in most cases the "detuning" of the mode locking due to a change of the cavity mode spacing caused by changes in r is negligible. We start with the assumption that the bandwidth of the mode locked pulse is a function of time. In (10.5) we replace ω_M^2/r_o by ω_I^2/r_o where ω_I is a function of time. This assumed expression for the current wave form obeys the eigenvalue equation

$$M\,\omega_m^2 \frac{d^2 I}{d\omega^2} - \frac{r_o \omega^2}{\omega_I^2} I = -\sqrt{Mr_o}\,\frac{\omega_m}{\omega_I}\, I \tag{10.11}$$

Now the time derivative of the assumed waveform is

$$\frac{dI}{dt} = \sqrt{T_R}\;\; v_o(\omega) \left[\mathring{A} + \frac{1}{2} A \left(\frac{\omega^2}{\omega_m\, \omega_I\, \sqrt{M/r_o}} - \frac{1}{2} \right) \frac{\mathring{\omega}_I}{\omega_I} \right] \tag{10.12}$$

The time derivative of the mode locked pulse contains a hermite polynomial of second order. This is the consequence of the fact that a changing width (in frequency space) of the pulse requires

growth of the wings at the expense of the center. Replacing the right-hand side of (10.2) by the time derivative, utilizing (10.12) and balancing the ω independent terms and the terms proportional to ω^2 one obtains

$$\dot{A} - \frac{1}{4} A \frac{\dot{\omega}_I}{\omega_I} = \frac{A}{x_c'} \left\{ r - 1 - \sqrt{M r_o} \frac{\omega_m}{\omega_I} \right\} \tag{10.13}$$

$$\frac{\dot{\omega}_I}{\omega_I} = - \frac{2\sqrt{M/r_o}}{x_c'} \frac{\omega_m}{\omega_I} \left[\frac{r \omega_I^2}{\omega_M^2} - r_o \right] \tag{10.14}$$

with $r = r_o / \left(1 + \frac{A^2}{P_s}\right)$

These are two coupled first order differential equations in the variables ω_I and A. Since they are nonlinear, they require in general a computer solution.

We shall derive here approximate solutions to these two equations. Multiplication of Eq. 10.14 by A so as to obtain an equation for the variable $A^2 = P$, subsequent introduction of the van der Pol approximation leads to

$$\frac{d}{dt}\left(\frac{P}{P_s}\right) = \frac{2}{x_c'} \left[r_o - 1 - r_o \frac{P}{P_s} - \sqrt{M} \frac{\omega_m}{\omega_I} \right] - \frac{P}{P_s} \frac{\sqrt{M r_o}}{x_c'} \frac{\omega_m}{\omega_I} \left[\frac{\omega_I^2}{\omega_M^2} - 1 \right] \tag{10.15}$$

One may expand the solutions of (10.14) and (10.15) in terms of a parameter of smallness $\sqrt{M}\ \omega_m/\omega_M$. The bandwidth itself, as a function of time, varies more slowly with time than the power and hence to zero order one may disregard the contribution of bandwidth variation in (10.15). Therefore, if one disregards terms of the order of $\sqrt{M}\ \omega_m/\omega_M$ one obtains Eq. (10.7). One may reintroduce the time dependence of the power into the linearized form of the equation for frequency (10.14). Setting $1/[1 + (P/P_s)] \simeq 1 - (P/P_s)$ and disregarding the product δP:

$$\dot{\delta} = -\frac{2\sqrt{Mr_o}}{x_c'}\ \frac{\omega_m}{\omega_M}\left\{2\delta - \frac{P}{P_s}\right\} \tag{10.16}$$

where

$$\delta \equiv \left(\frac{\omega_I}{\omega_M} - 1\right) \tag{10.17}$$

Integration gives

$$\delta = e^{-at}\ \frac{\alpha_g}{\beta_g}\int_{-\infty}^{t} \frac{1}{2}\ adt\ \frac{e^{(2\alpha_g + a)t}}{1 + e^{2\alpha_g t}} \tag{10.18}$$

where

$$a \equiv \frac{4\sqrt{Mr_o}}{\omega_M}\left(\frac{1}{x_c'}\right)\omega_m$$

The effective bandwidth which is contained in the pulse formula (10.5) starts out at ω_M, the value prescribed by the line width in the absence of power, and approaches asymptotically the steady state value

$$\omega_I = \omega_M \left[\frac{1}{2} \frac{\alpha_g}{\beta_g} + 1 \right]$$

Hence, the bandwidth of the pulse increases as a function of time as the laser medium is saturation broadened. The bandwidth as a function of time is shown in Figure 9.

XI. Stability of Solutions of Homogeneously Broadened Laser Medium

In the subsequent sections we study the stability of the supermodes with respect to arbitrary perturbations. It is assumed that the mode under consideration, say the n-th supermode, is perturbed, the perturbation is expanded in the complete set of eigenfunctions of the differential equation (harmonic oscillator equation). Growth of any one component of the perturbation signifies instability.

We shall follow the analysis of stability of Kurokawa. In this analysis, the perturbation is assumed to be a slow function of time (compared with the time variation of the unperturbed oscillation). This condition is clearly met in most laser systems.

In the same way as we treated the buildup of a supermode, we treat the time evolution of a perturbation in terms of its complex frequency $\delta\omega$, where $\mathrm{Im}\,\delta\omega$ is the rate of decay (if positive) of the perturbation, in general a function of ω (i.e. $k\omega_m$ and hence mode number). Because we assume that the perturbation $\delta I(\omega)$ is in general complex and time dependent, frequencies of oscillation of the perturbed modes are allowed to deviate from the steady state frequency. In other words, we treat the most general time dependent perturbation with the only restriction that the time rate of change of the perturbation be slow compared to the cavity mode bandwidth. Taking a perturbation of (10.2), and taking into account that $j\delta\omega\, x_c'\, \delta I = x_c'\, d/dt\, \delta I$ and assuming a resonant modulation, we obtain

$$\left[M\,\omega_m^{\,2}\,\frac{d^2}{d\omega^2} + r\left(1 - \frac{\omega^2}{\omega_M^{\,2}}\right)\right]\delta I$$

$$+ (\delta r)\left(1 - \frac{\omega^2}{\omega_M^{\,2}} - j\,\frac{\omega}{\omega_M}\right) I_n = x_c{}'\,\frac{d}{dt}\,\delta I \qquad (11.1)$$

Here we have denoted the unperturbed solution by the subscript n, assuming that the n-th supermode exists within the cavity. Next, consider the perturbation $I_n \delta r$.

$$I_n \delta r = I_n\, r_o\, \delta\left(\frac{1}{1 + \frac{P}{P_s}}\right) = \frac{I_n r_o}{\left(1 + \frac{P}{P_s}\right)^2}\,\delta\left(\frac{P}{P_s}\right) \qquad (11.2)$$

If the current $I(\omega)$ is so normalized that $\int |I(\omega)|^2\, d\omega$ is equal to the power, then

$$\delta P = \int |I_n(\omega) + \delta I(\omega)|^2\, d\omega - \int |I_n(\omega)|^2\, d\omega = 2 \int I_n(\omega)\ \mathrm{Re}\ \delta I\, d\omega \qquad (11.3)$$

It is convenient to express δI as a superposition of the complete set of eigenfunctions of the harmonic oscillator equation:

$$\left[M\,\omega_m^{\,2}\,\frac{d^2}{d\omega^2} - r\,\frac{\omega^2}{\omega_M^{\,2}}\right] u = -\,E\,u \qquad (11.4)$$

The eigenvalues of these equations are (compare 4.9)

$$E_n = 2\sqrt{Mr}\ \frac{\omega_m}{\omega_M}\left(n + \frac{1}{2}\right) \tag{11.5}$$

The eigenfunctions are identical with the supermodes (4.8). We make them all real and normalize them so that $\int u_n\, u_m\, d\omega = \delta_{nm}$ (11.6)
We get

$$\delta I = \sum_m a_m(t)\ u_m(\omega) \tag{11.7}$$

and introduce this expression into (11.1). We obtain

$$-\sum_m (E_m - E_n)\ a_m\ u_m\ (\omega) - \frac{r\,\frac{P}{P_s}}{1 + \frac{P}{P_s}}\ 2\ \mathrm{Re}\ a_n\ u_n\ (\omega)\left(1 - \frac{\omega^2}{\omega_M^{\ 2}} - j\,\frac{\omega}{\omega_M}\right)$$

$$= x_c{}'\ \frac{d}{dt}\ \Sigma\ a_m\ u_m\ (\omega)$$

We multiply both sides by $u_m(\omega)$, integrate over all frequencies, and use the orthonomality relation (11.6). The result is

$$-\ (E_m - E_n)\ a_m - \frac{r\,\frac{P}{P_s}}{1 + \frac{P}{P_s}}\ 2\ \mathrm{Re}\ a_n \int u_n(\omega)\left(1 - \frac{\omega^2}{\omega_M^{\ 2}} - j\,\frac{\omega}{\omega_M}\right)u_m(\omega)\,d\omega = x_c{}'\ \frac{d}{dt}\ a_m \tag{10.18}$$

The term under the integral couples m and n if they differ by ± 1 or ± 2. Concentrating on $m = n - 1$, and studying only the real part of (11.8) we find

$$-(E_{n-1} - E_n)\,\mathrm{Re}\, a_{n-1} = x_c' \frac{d}{dt} \mathrm{Re}\, a_{n-1}$$

which gives exponential growth because $E_{n-1} < E_n$. The n-th supermode is unstable with respect to a perturbation proportional to the mode of the next lower order.

Next, set $n = m = 0$ and study the resulting equation. One has one equation for the imaginary part of a_o, another one for the real part

$$\frac{d}{dt} \mathrm{Im}\, a_o = 0$$

$$\frac{d}{dt} \mathrm{Re}\, a_o = - \frac{2r \dfrac{P}{P_s}}{1 + \dfrac{P}{P_s}} \left(1 - \frac{1}{2} \frac{\omega_p^2}{\omega_M^2}\right) \mathrm{Re}\, a_o \qquad (11.9)$$

where we have used the fact that

$$\int u_o^2(\omega) \left(1 - \frac{\omega^2}{\omega_M^2} - j \frac{\omega}{\omega_M}\right) d\omega = 1 - \frac{1}{2} \frac{\omega_p^2}{\omega_M^2} \qquad (11.10)$$

with

$$\omega_p^2 = \sqrt{M/r}\, \omega_M\, \omega_m \qquad (11.11)$$

The component $\mathrm{Re}\, a_o$ decays at the rate

$$\frac{1}{\tau} = \frac{r}{1 + \dfrac{P}{P_s}}\, 2 \left(1 - \frac{1}{2} \frac{\omega_p^2}{\omega_M^2}\right) \frac{P}{P_s} \qquad (11.12)$$

This decay rate is always positive, even though (11.12) seems to permit negative values. We recall, however, that we have expanded a Lorentzian, an approximation valid only when the pulse bandwidth ω_p is small compared with ω_M. If the expansion had not been made, (11.10) would be positive definite.

The component $\mathrm{Im}\, a_0$ has no growth or decay. Hence a quadrature perturbation $a_0\, u_0(\omega)$ experiences no growth or decay: i.e. there is no restoring force for a perturbation of this kind. A quadrature perturbation corresponds to a phase perturbation of the carrier frequency. Hence, the mode locked pulses have no carrier phase stabilization, just like a free running van der Pol oscillator[8].

XII. The stability of Supermodes for Inhomogeneous Laser Medium

Whereas closed form solutions do not exist for the inhomogeneously broadened medium, it is still possible to carry out a stability analysis of the supermodes whose qualitative features have been determined in Section IX. The differential equation for the steady state solution for resonant mode locking is (compare 9.1)

$$M\,\omega_m^2\,\frac{d^2 I}{d\omega^2} = \left[1 - \frac{r_o(1 - \omega^2/\omega_M^2)}{1 + \dfrac{A^2(\omega)}{A_s^2}}\right] I \tag{12.1}$$

The situation in the present case is simpler than for the homogeneously broadened medium, because we disregard the reactance associated with the laser medium. We write down the equation governing a perturbation δI, as an obvious extension of the preceding section. Assuming that the steady state is the n-th supermode:

$$M\,\omega_m^2\,\frac{d^2}{d\omega^2}\,\delta I + \left[\frac{r_o}{1 + \dfrac{A_n^2(\omega)}{A_s^2}}\left(1 - \frac{\omega^2}{\omega_M^2}\right) - 1\right]\delta I \tag{12.2}$$

$$- \left[\frac{r_o}{\left[1 + \dfrac{A_n^2(\omega)}{A_s^2}\right]^2}\left(1 - \frac{\omega^2}{\omega_M^2}\right)\frac{2\,A_n^2(\omega)}{A_s^2}\right] \mathrm{Re}\,\delta I = x_c'\,\frac{d}{dt}\,\delta I$$

We may separate this equation into real and imaginary parts, obtaining for the imaginary part

$$\left\{ M\,\omega_m^2 \frac{d^2}{d\omega^2} + \left[\frac{r_o}{1 + \dfrac{A_n^2(\omega)}{P_s}} \left(1 - \frac{\omega^2}{\omega_M^2}\right) - 1 \right] \right\} \mathrm{Im}\,\delta I = \frac{d}{dt}\,\mathrm{Im}\,\delta I \tag{12.3}$$

This equation already can be used to demonstrate that all higher order supermodes are unstable. Indeed, let us expand the perturbation

$$\mathrm{Im}\,\delta I = \sum_p a_p\, u_p(\omega) \tag{12.4}$$

where the $u_p(\omega)$ are the orthonormal eigenfunctions of the differential equation

$$-\left\{ M\,\omega_m^2 \frac{d^2}{d\omega^2} + \frac{r_o}{1 + \dfrac{A_n^2(\omega)}{A_s^2}} \left(1 - \frac{\omega^2}{\omega_M^2}\right) \right\} u_m(\omega) = E_m\, u_m(\omega) \tag{12.5}$$

By definition $E_n = -1$ for the eigenfunction corresponding to the steady state solution, which itself is an eigenfunction of (12.5). Now it is clear from the nature of the problem that $E_m < -1$ for an eigenfunction with less peaks (less kinetic energy and hence less negative curvature) than the assumed unperturbed solution. Introducing (12.4) into (12.3), multiplying the

equation by $u_m(\omega)$, integrating over all ω and using the orthonormality condition, one obtains

$$-(1 + E_m)\, a_m = x_c' \frac{d}{dt}\, a_m \tag{12.6}$$

Equation (12.6) shows that a_m grows exponentially for an eigenfunction m for which $E_m < -1$, i.e. for an eigenfunction with less peaks than the lowest order eigenfunction. Hence, all supermodes are unstable except (maybe) the one with a single peak. We turn to the analysis of this mode, which we denote by the subscript 0. Consider (12.6) applied to a perturbation proportional to the lowest order eigenmode, a_o. Since $E_o = -1$, we find

$$\frac{d}{dt}\, \mathrm{Im}\, a_o = 0$$

The perturbation experiences no "restoring force". This is entirely analogous to the homogeneously broadened case in which quadrature perturbations also did not experience restoring forces.

Now consider the equation for the real part of the amplitude

$$\Bigg\{ M\, \omega_m^2 \frac{d^2}{d\omega^2} + \frac{r_o}{1 + \dfrac{A_o^2(\omega)}{A_s^2}} \left(1 - \frac{\omega^2}{\omega_M^2}\right) - 1$$

$$- \frac{2r_o}{\left[1 + \dfrac{A_o^2(\omega)}{A_s^2}\right]^2} \left(1 - \frac{\omega^2}{\omega_M^2}\right) \frac{A_o^2(\omega)}{A_s^2} \Bigg\}\, \mathrm{Re}\, \delta I = x_c' \frac{d}{dt}\, \mathrm{Re}\, \delta I \tag{12.7}$$

We may expand $\mathrm{Re}\,\delta I$ in terms of the eigenfunctions u_n of the differential equation

$$-\left\{M\,\omega_m^{\,2}\frac{d^2}{d\omega^2}+\frac{r_o}{1+\dfrac{A_o^{\,2}(\omega)}{A_s^{\,2}}}\left[1-\frac{2}{1+\dfrac{A_o^{\,2}(\omega)}{A_s^{\,2}}}\;\frac{A_o^{\,2}(\omega)}{A_s^{\,2}}\right]\left(1-\frac{\omega^2}{\omega_M^{\,2}}\right)\right\}u_n(\omega)$$

$$= E_n\,u_n(\omega) \tag{12.8}$$

This set of eigenfunctions does not contain the unperturbed steady state solution because of the additional term in parentheses. It is clear however that the eigenfunction of (12.8) with a single peak denoted $u_o(\omega)$ possesses the lowest eigenvalue and hence is the most "dangerous" so far as growth is concerned.

If the term in the brackets containing $A_o^{\,2}(\omega)$ were missing, $u_o(\omega)$ would reduce to the steady state solution, and E_o would be equal to -1.

It is clear that

$$-\left(1-\frac{\omega^2}{\omega_M^{\,2}}\right)\frac{r_o}{1+\dfrac{A^2(\omega)}{A_s^{\,2}}}$$

by itself represents the potential well of the steady state problem . The additional term modifies the well. The deeper the well, the lower the eigenvalue, the higher the well, the higher the eigen-

value. It is clear that the additional term raises the well and hence raises the eigenvalue. Therefore

$$E_o > -1.$$

Returning to (12.8), introducing

$$\mathrm{Re}\ \delta I = \sum_p a_p u_p(\omega)$$

multiplying both sides of (12.8) by $v_o(\omega)$ and integrating over all ω, we obtain

$$-(1 + E_o)\, a_o = x_c' \frac{d}{dt} a_o \qquad (12.9)$$

and since $E_o > -1$, we find that the perturbation $a_o u_o(\omega)$ decays. Perturbations proportional to higher order eigenfunctions decay even faster. In this way we have proven the stability of the lowest order supermode of the inhomogeneously broadened case.

XIII. Conclusions

The replacement of the discrete cavity spectrum by a continuum made the treatment of mode locking in the frequency domain a relatively simple mathematical problem which gives physical intuition a free reign. We were able to find all the steady state solutions of the homogeneously broadened laser medium, but found that only the lowest order solution was a stable one.

The inhomogeneously broadened laser medium does not have modes which can be evaluated in closed form, yet their physical properties were easily deduced from the particle motion analog. That this kind of approach is a useful one was confirmed by the stability analysis of the supermode of the inhomogeneously broadened laser for the execution of which it was not necessary to find the detailed functional dependence upon frequency of the supermode. Again it was found that the only stable supermode was the one with a single peak when plotted versus frequency.

The effect of detuning was obtained. The build-up of the stable supermode, as a function of time, was evaluated and the attendant change in bandwidth of the transient supermode was determined for the case of a homogeneously broadened laser medium.

It is believed that the potential of the present analysis has been hardly tapped and that many other issues of interest, both in forced mode locking and saturable absorber mode locking, will be analyzable using this new approach.

FIGURE CAPTIONS

1. Equivalent circuit for single mode, injection locked oscillator.

2. The equivalent circuit for multimode injection locked oscillator system.

3. Eigensolutions (Supermodes) of the homogeneously broadened laser in the frequency domain.

4. The force-field-potential as a function of "time".

5. The normalized amplitude versus normalized frequency of the lowest order supermode for an inhomogeneously broadened laser.

6. Range of normalized gain and normalized mode locking modulation within which steady state mode locking solutions with single peak are found.

7. Amplitude at line center as function of normalized modulation.

8. Transient build-up of normalized power in the cavity.

9. Bandwidth as a function of time.

Acknowledgment

The author is grateful to Mr. Wallace Wong, who obtained the computer solution of Figures 5, 6, and 7. He also acknowledges the many useful discussions with Mr. C. Ausschnitt.

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Figure 1.

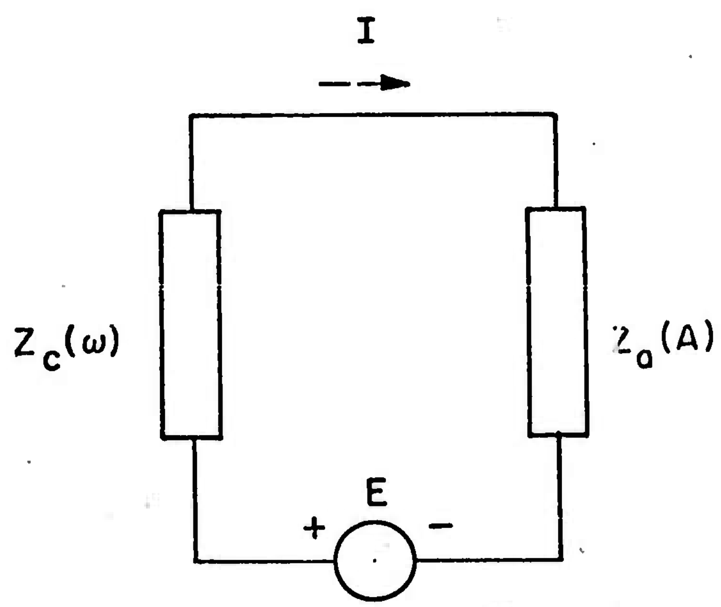
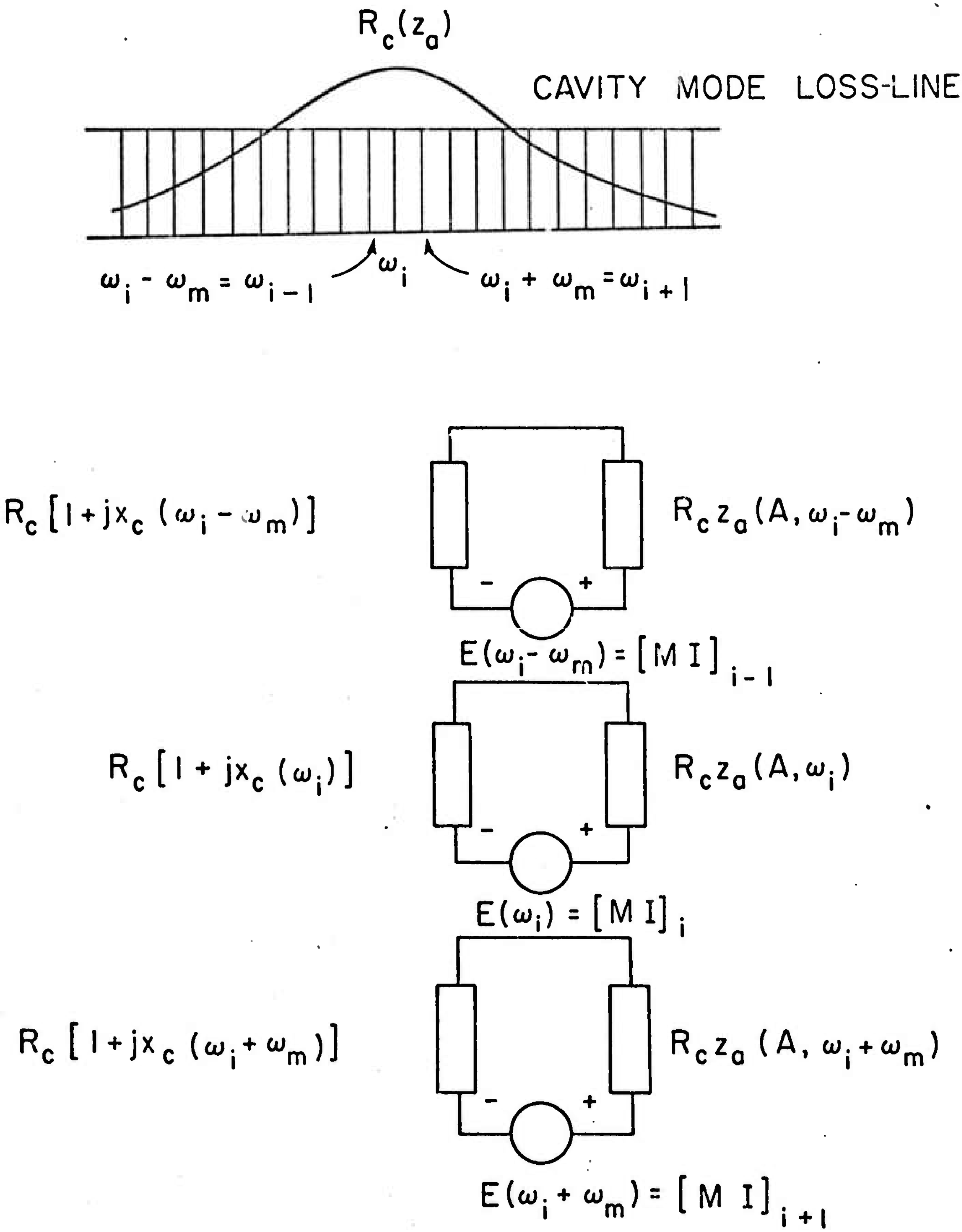


Figure 2.



$R_c(z_a)$
CAVITY MODE LOSS-LINE
$\omega_i - \omega_m = \omega_{i-1}$
ω_i
$\omega_i + \omega_m = \omega_{i+1}$
$R_c[1 + jx_c(\omega_i - \omega_m)]$
$R_c z_a(A, \omega_i - \omega_m)$
$E(\omega_i - \omega_m) = [M\,I]_{i-1}$
$R_c[1 + jx_c(\omega_i)]$
$R_c z_a(A, \omega_i)$
$E(\omega_i) = [M\,I]_i$
$R_c[1 + jx_c(\omega_i + \omega_m)]$
$R_c z_a(A, \omega_i + \omega_m)$
$E(\omega_i + \omega_m) = [M\,I]_{i+1}$

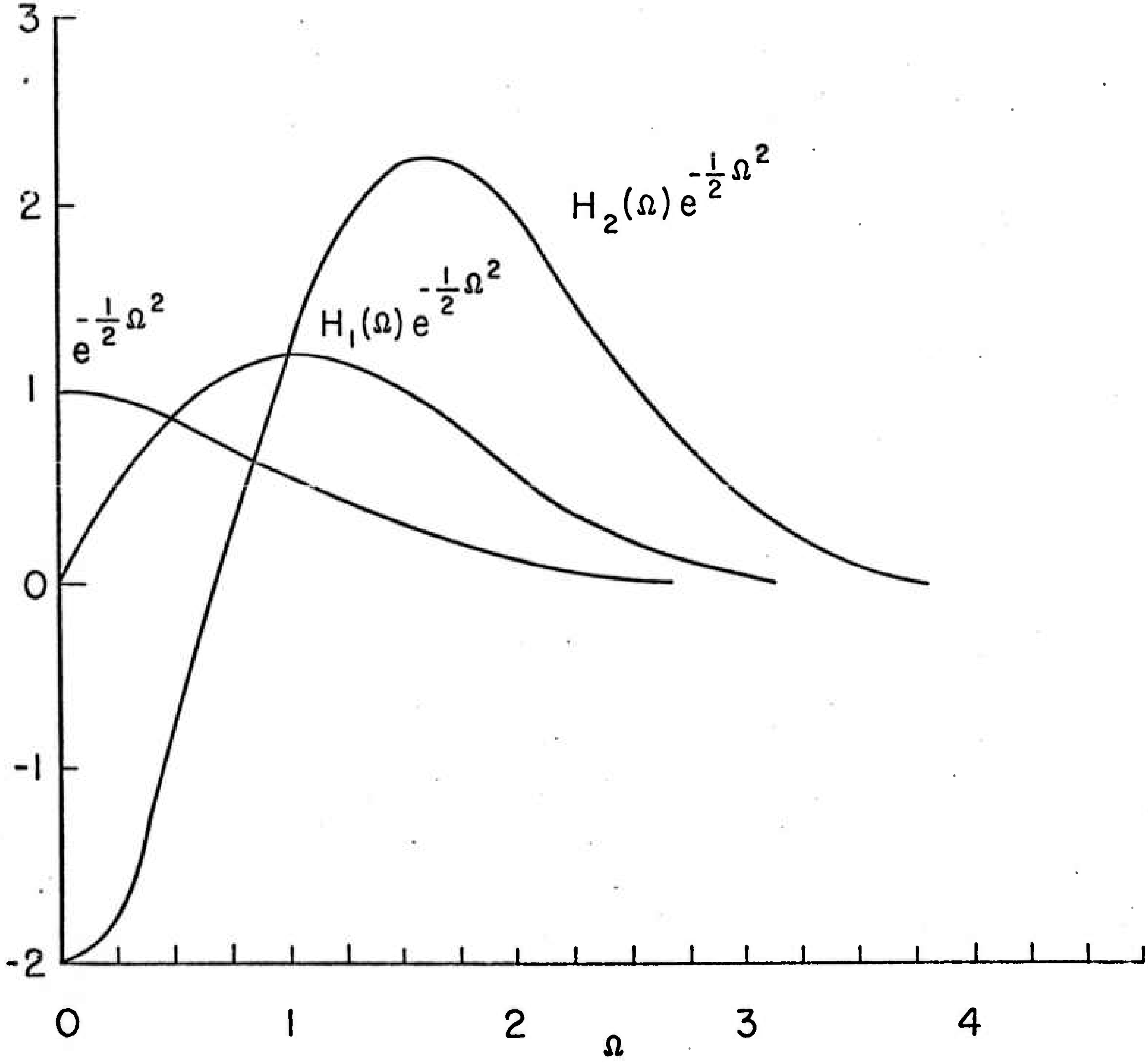
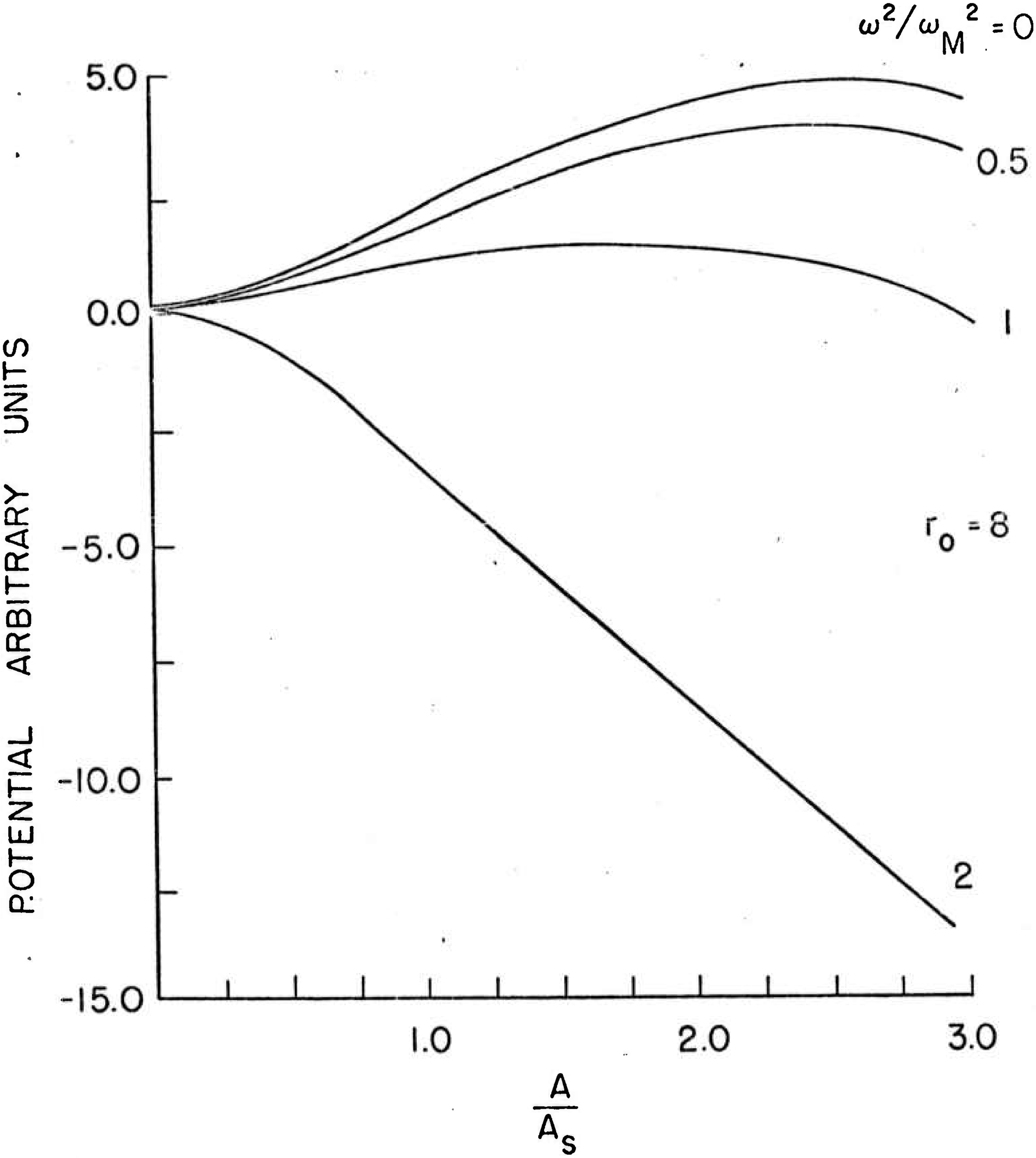
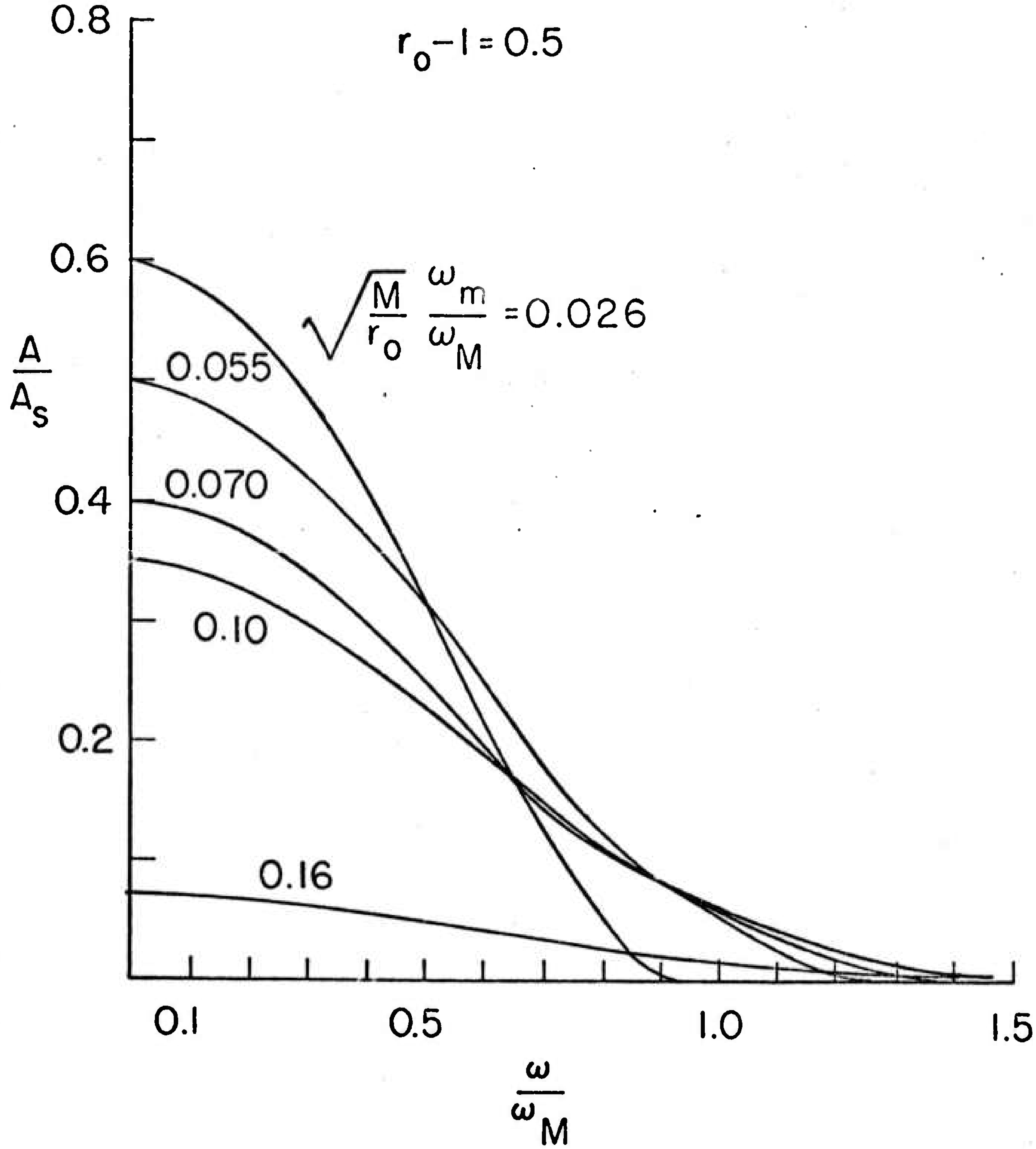
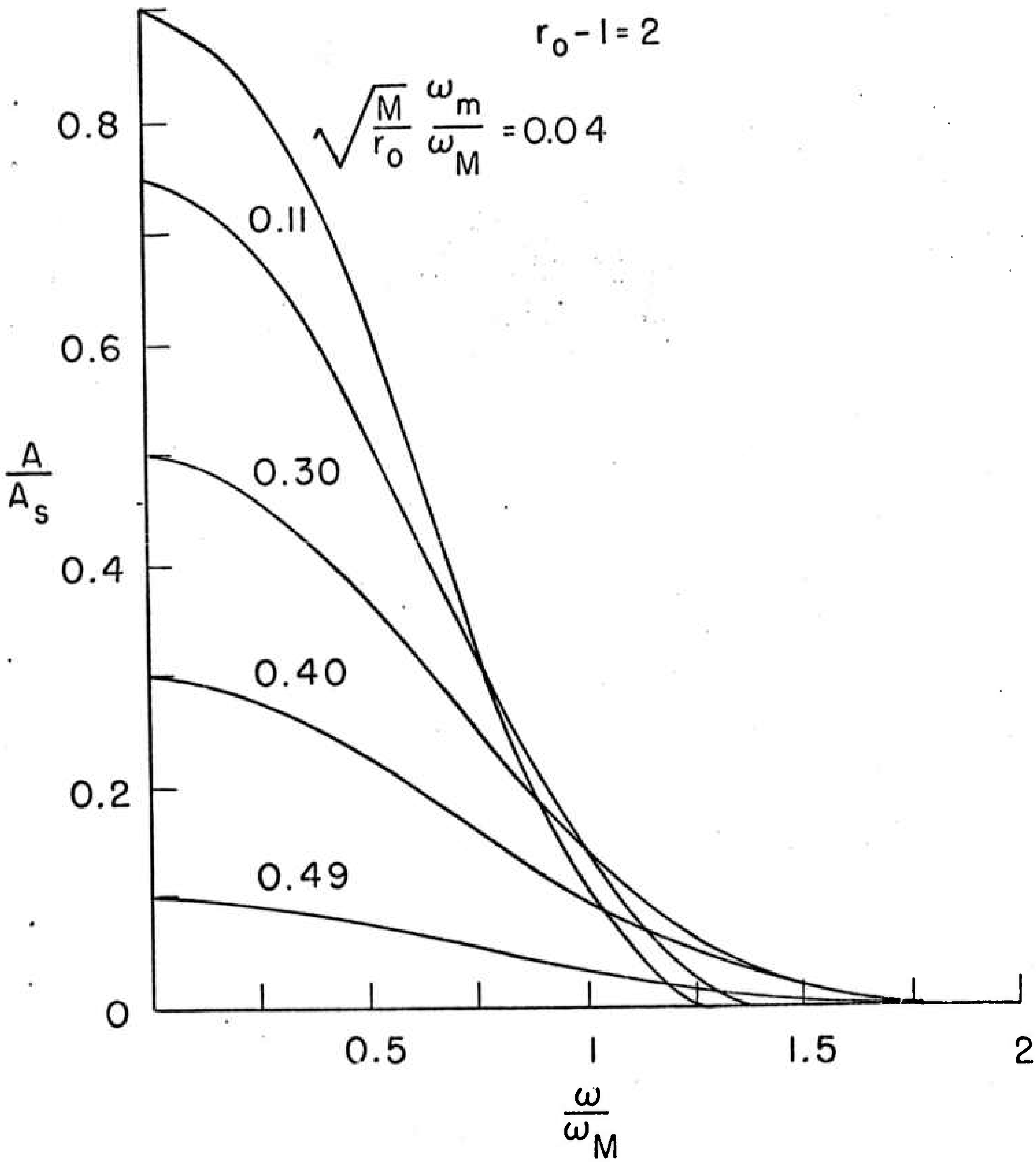

3
2
1
0
-1
-2
0
1
2
3
4
Ω
$e^{-\frac{1}{2}\Omega^2}$
$H_1(\Omega)\, e^{-\frac{1}{2}\Omega^2}$
$H_2(\Omega)\, e^{-\frac{1}{2}\Omega^2}$

Figure 4.



$\omega^2/\omega_M^2 = 0$
0.5
1
2
$r_o = 8$
5.0
0.0
-5.0
-10.0
-15.0
POTENTIAL ARBITRARY UNITS
1.0
2.0
3.0
$\frac{A}{A_s}$


0.8
0.6
0.4
0.2
$r_o - 1 = 0.5$
$\sqrt{\frac{M}{r_o}}\ \frac{\omega_m}{\omega_M} = 0.026$
0.055
0.070
0.10
0.16
$\frac{A}{A_s}$
0.1
0.5
1.0
1.5
$\frac{\omega}{\omega_M}$


$r_0 - 1 = 2$
$\sqrt{\frac{M}{r_0}} \frac{\omega_m}{\omega_M} = 0.04$
0.11
0.30
0.40
0.49
$\frac{A}{A_s}$
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
$\frac{\omega}{\omega_M}$

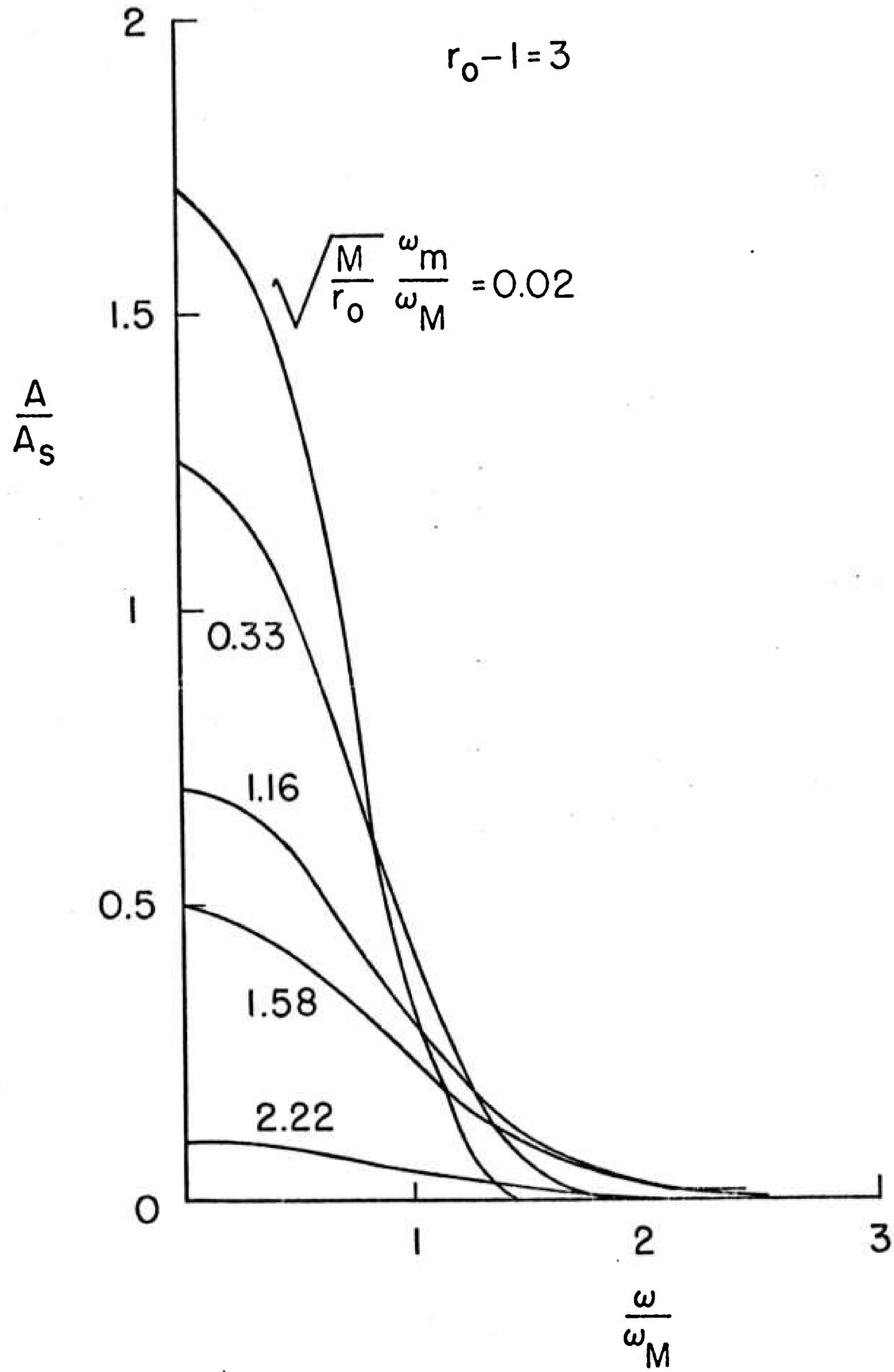
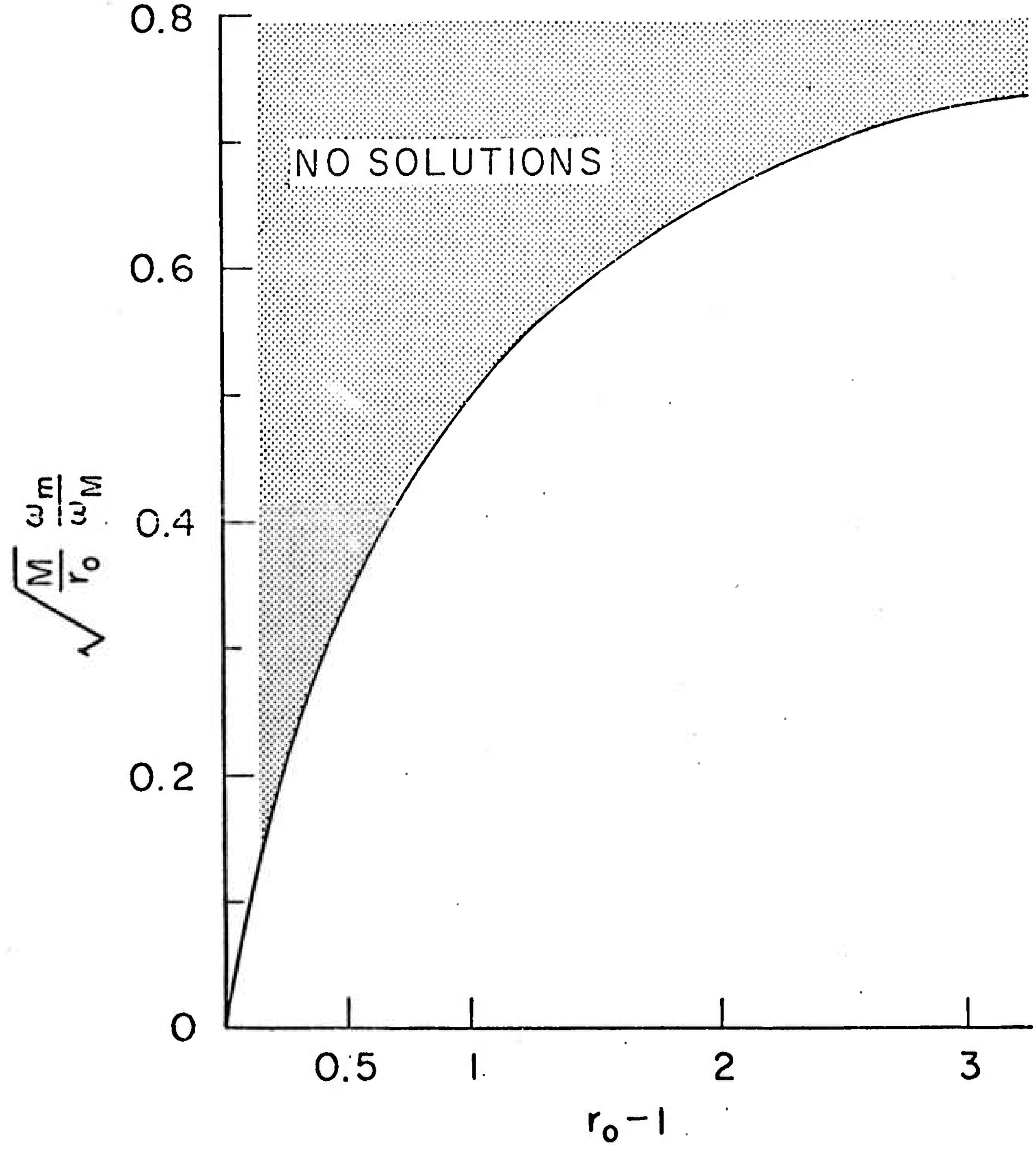
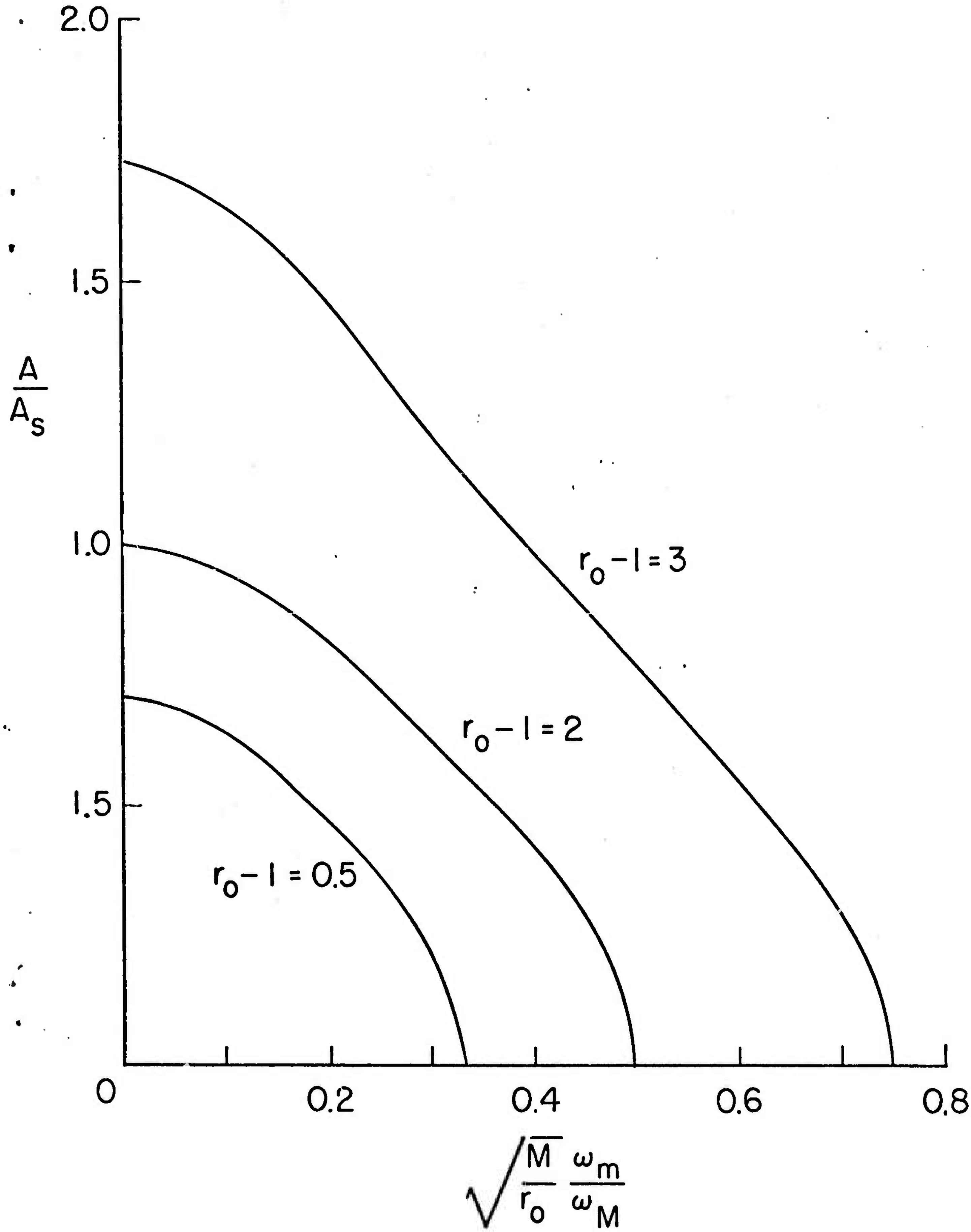

$r_o - 1 = 3$
$\sqrt{\frac{M}{r_o}}\frac{\omega_m}{\omega_M} = 0.02$
0.33
1.16
1.58
2.22
$\frac{A}{A_s}$
$\frac{\omega}{\omega_M}$
0
0.5
1
1.5
2
1
2
3

Figure 6.



NO SOLUTIONS
0.8
0.6
0.4
0.2
0
0.5
1
2
3
$r_o - 1$
$\sqrt{\frac{M}{r_o}}\,\frac{\omega_M}{\omega_m}$

Figure 7.



2.0
1.5
1.0
1.5
$\frac{A}{A_s}$
$r_o - 1 = 3$
$r_o - 1 = 2$
$r_o - 1 = 0.5$
0
0.2
0.4
0.6
0.8
$\sqrt{\frac{M}{r_o}}\,\frac{\omega_m}{\omega_M}$

Figure 8.

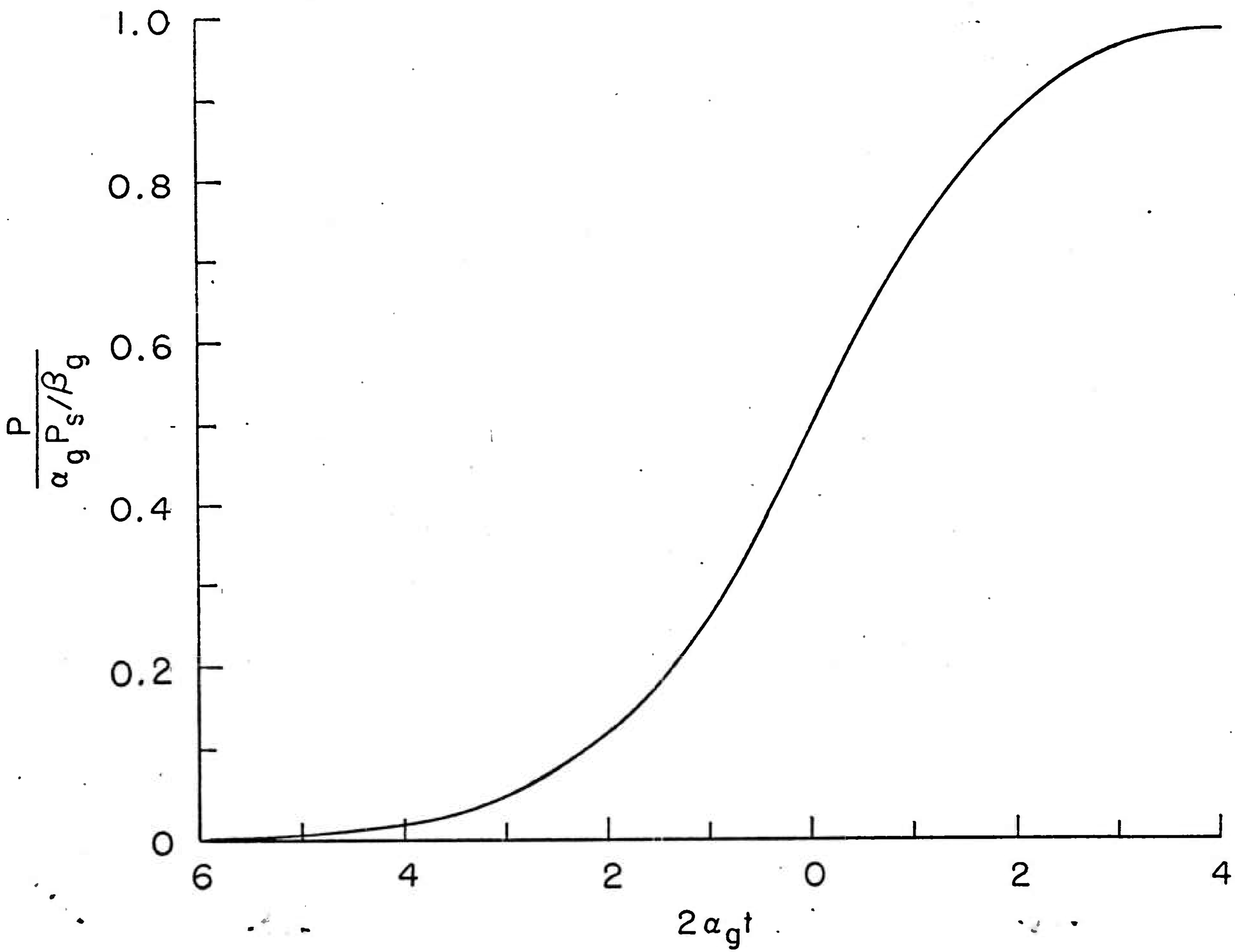
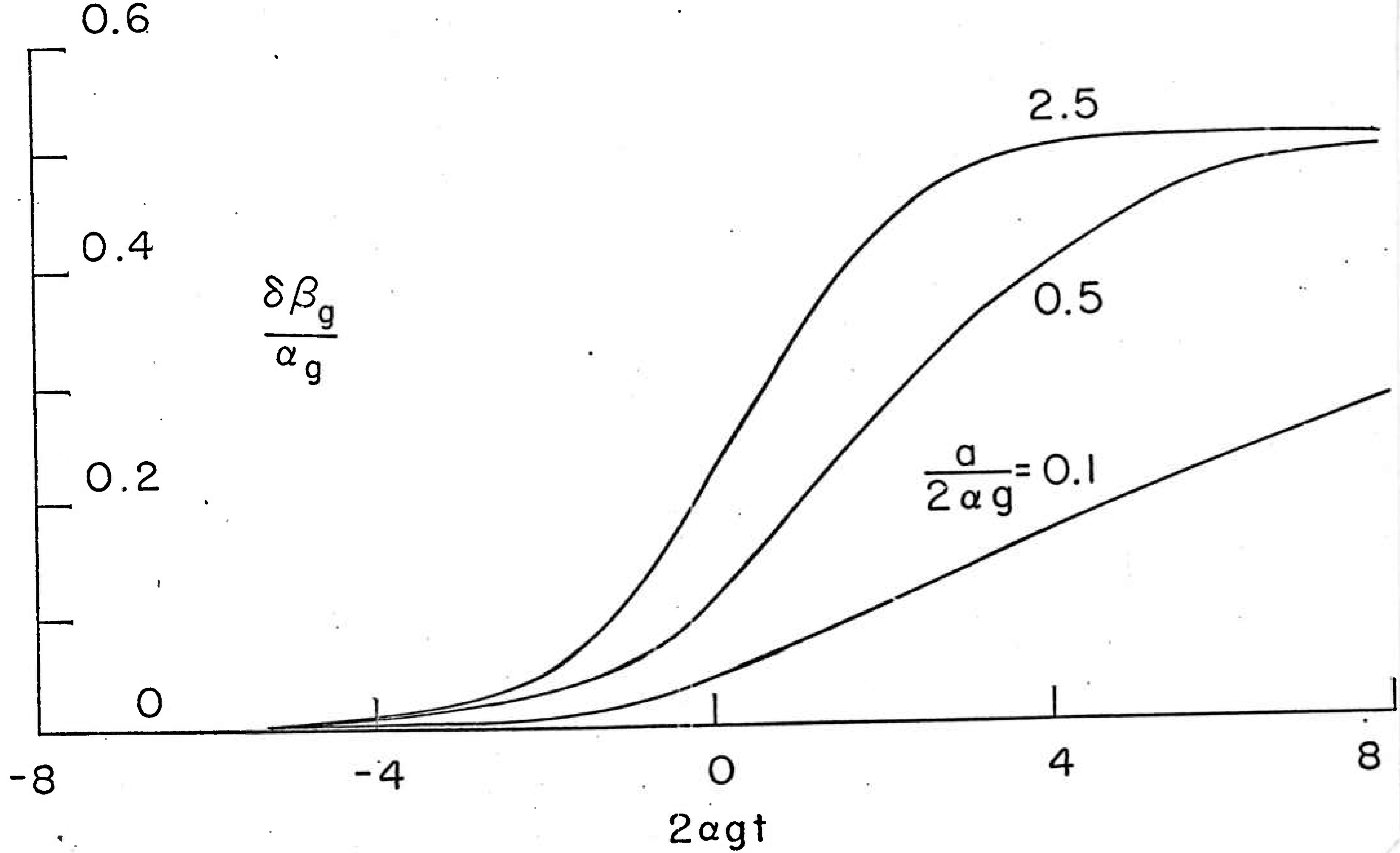


Figure 9.


0.6
0.4
0.2
0
$\frac{\delta\beta_g}{\alpha_g}$
2.5
0.5
$\frac{a}{2\alpha g}=0.1$
-8
-4
0
4
8
$2\alpha gt$